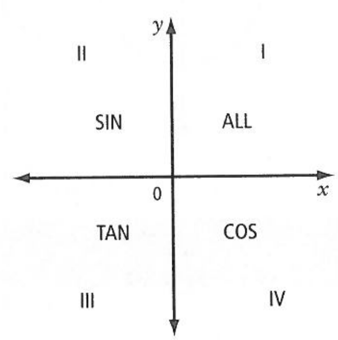
4.4 Introduction to Trigonometric Equations

KEY IDEAS

- Solving an equation means to determine the value (or values) of a variable that make an equation true (Left Side = Right Side).
  For example, $\sin \theta = \frac{1}{2}$ is true when $\theta = 30°$ or $\theta = 150°$, and for every angle coterminal with 30° or 150°. These angles are solutions to a very simple trigonometric equation.
- The variable θ is often used to represent the unknown angle, but any other variable is allowed.
- In general, solve for the trigonometric ratio, and then determine
  – all solutions within a given domain, such as $0 \leq \theta < 2\pi$
  or
  – all possible solutions, expressed in general form, $\theta + 2\pi n$, $n \in I$
- Unless the angle is a multiple of 90° or $\frac{\pi}{2}$, there will be two angles per solution of the equation within each full rotation of 360° or 2π. As well, there will be two expressions in general form per solution, one for each angle. It is sometimes possible to write a combined expression representing both angles in general form.
- If the angle is a multiple of 90° or $\frac{\pi}{2}$ (that is, the terminal arm coincides with an axis), then there will be at least one angle within each full rotation that is a correct solution to the equation.
- Note that $\sin^2 \theta = (\sin \theta)^2$. Also, recall that
  – $\sin \theta$ and $\csc \theta$ are positive in quadrants I and II
  – $\cos \theta$ and $\sec \theta$ are positive in quadrants I and IV
  – $\tan \theta$ and $\cot \theta$ are positive in quadrants I and III



Working Example 1: Solve a First-Degree Trigonometric Equation

Solve for the angle (in degrees). Round answers to the nearest tenth of a degree.

a) $7 \cos \theta + 5 = 2 - 3 \cos \theta$, $0° \le \theta < 360°$

b) $17 + 3 \cot \theta = 29$, in general form

Solution

a) Start by solving for the trigonometric ratio.

$7 \cos \theta + 5 = 2 - 3 \cos \theta$

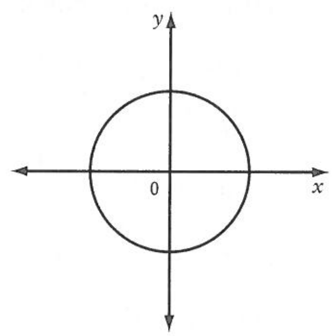
> Treat $\cos \theta$ as a variable and isolate it on one side of the equation.

$\cos \theta =$ ________

Determine the reference angle.

$\theta_R = \cos^{-1}(+0.3)$

$\approx$ ________



Cosine is negative in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant II angle: ________

quadrant III angle: ________

Therefore, the solutions are $\theta =$ ________ and ________, $0° \le \theta < 360°$.

b) Start by solving for the trigonometric ratio.

$17 + 3 \cot \theta = 29$

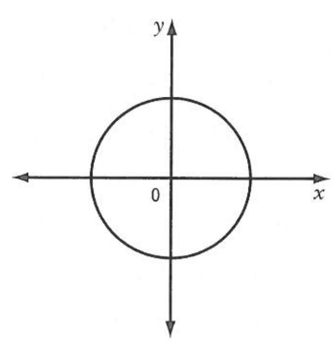
$3 \cot \theta =$ ________

$\cot \theta =$ ________

$\cot \theta$ is the reciprocal of ________. Therefore, $\tan \theta =$ ________.

Determine the reference angle.

$\theta_R =$ ________



Tangent and cotangent are positive in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

$\theta_1 =$ ________ $\theta_2 =$ ________

Can θ_1 and θ_2 be generalized in a single expression?

Notice on your diagram that $\theta_2 = \theta_1 + 180°$. There is a solution every 180°.

The general form is ________________ $+ 180°n$, $n \in I$.

Working Example 2: Solve Second-Degree Equations

Solve for the unknown value. If necessary, round your answer to two decimal places.

a) $2 \sin^2 \theta = 1, 0 \leq \theta < 2\pi$

b) $\tan^2 \theta - 4 \tan \theta + 3 = 0, 0 \leq \theta < 2\pi$

Solution

a) Isolate the trigonometric ratio $\sin^2 \theta$. Then, take the square root of both sides.

$2 \sin^2 \theta = 1$

$\sin^2 \theta =$ _______________

$\sin \theta =$ ________ or $\sin \theta =$ ________

> What is the degree of the equation? How many solutions are there?

Determine the reference angles for both solutions.

$\theta_R =$ ________ $\theta_R =$ ________

sin > 0 (positive) in Q ________ and Q ________

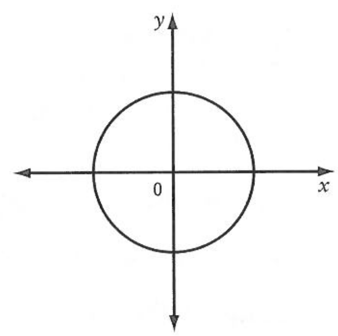
$\theta_1 =$ ________ $\theta_2 =$ ________

or

sin < 0 (negative) in Q ________ and Q ________

$\theta_3 =$ ________ $\theta_4 =$ ________

Therefore, the solutions are __________________________,
$0 \leq \theta < 2\pi$.



b) Factor and solve for $\tan \theta$.

$$\tan^2 \theta - 4 \tan \theta + 3 = 0$$

$(\tan \theta -$ ________$)(\tan \theta -$ ________$) = 0$

> Treat $\tan \theta$ as a variable.

Determine the reference angles for both solutions.

> Is your calculator in radians?

$\theta_R =$ ________ $\theta_R =$ ________

tan > 0 (positive) in Q ________ and Q ________

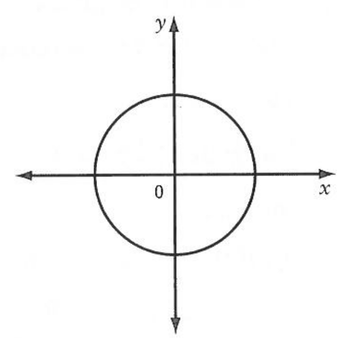
$\theta_1 =$ ________ $\theta_2 =$ ________

or

tan < 0 (negative) in Q ________ and Q ________

$\theta_3 =$ ________ $\theta_4 =$ ________

Therefore, the solutions are __________________________,
$0 \leq \theta < 2\pi$.



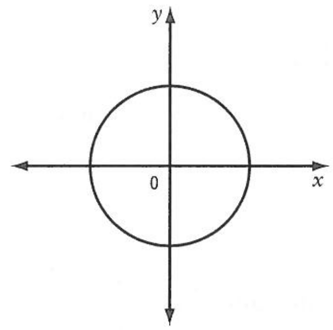
Also see Example 2 on page 208–209 of *Pre-Calculus 12*.

Check Your Understanding

Practise

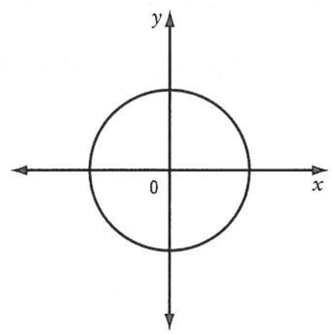
1. Determine the exact solutions for each trigonometric equation in the specified domain.

   a) $4 \sin \theta - 5 = 3, 0° \leq \theta < 360°$



   b) $7 \cot \theta - 4 = 6 \cot \theta - 5, 0 \leq \theta < 4\pi$

   $\cot \theta$ is the reciprocal of ________.

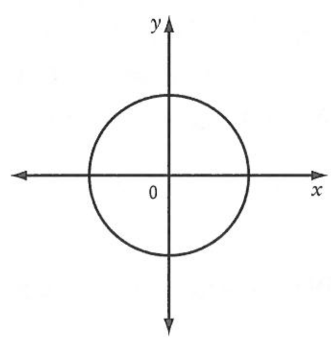


   Therefore, the solutions are

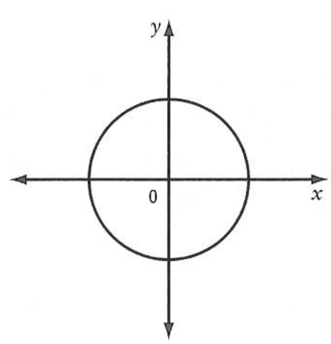
   $\theta =$ ______________________________, $0 \leq \theta < 4\pi$.

2. Solve for θ within the domain $0° \leq \theta < 360°$. Round answers to one decimal place.

   a) $-3(5 - 4 \sec \theta) = \sec \theta$



   b) $\csc \theta + \frac{3}{4} = -\frac{2}{3}$

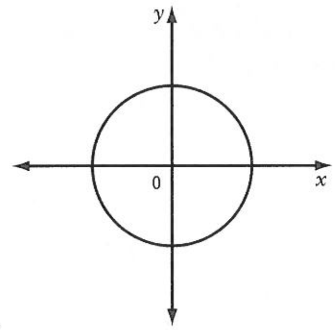


3. Solve for θ within the given domain. Give exact answers where possible. Otherwise, round your answer to two decimal places.

a) $4\cos^2\theta = 3, 0 \le \theta < 2\pi$

Treat cos θ as a variable.

Determine the reference angles for both solutions.

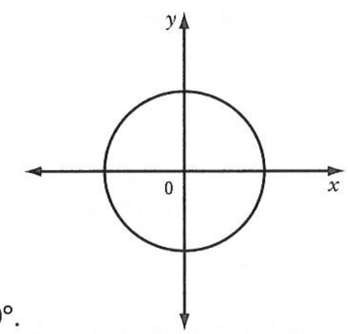


The solutions are ____________________, $0 \le \theta < 2\pi$.

b) $\csc^2\theta - 3\csc\theta - 10 = 0, 0 \le \theta < 360°$

Degrees or radians?

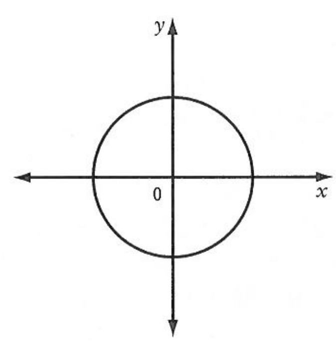
Determine the reference angles for both solutions.



The solutions are ____________________, $0 \le \theta < 360°$.

4. The equation $\cos\theta = \frac{\sqrt{3}}{2}, 0 \le \theta < 2\pi$, has solutions $\frac{\pi}{6}$ and $\frac{11\pi}{6}$. Suppose the domain is not restricted.

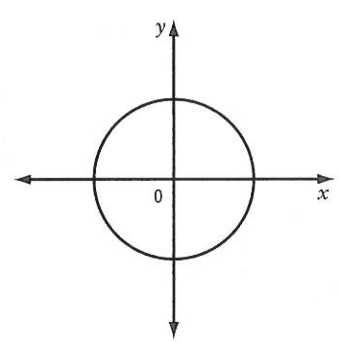
a) Write the general solution corresponding to $\frac{\pi}{6}$.



b) Write the general solution corresponding to $\frac{11\pi}{6}$.

5. The equation $\tan\theta = 1, 0 \le \theta < 2\pi$, has solutions $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

a) Write the solutions for $\tan\theta = 1$ if the domain is $0 \le \theta < 4\pi$.



b) Suppose the domain is unrestricted. Write the general solution.

Apply

6. The following solution contains one or more errors. Identify the errors and correct them.

Solve $\tan^2 \theta - 3 \tan \theta = 0$ in the domain $0 \leq \theta < 2\pi$.
Round answers to two decimal places.

$\tan^2 \theta - 3 \tan \theta = 0$

$\tan^2 \theta = 3 \tan \theta \quad (\div \tan \theta)$

$\tan \theta = 3$

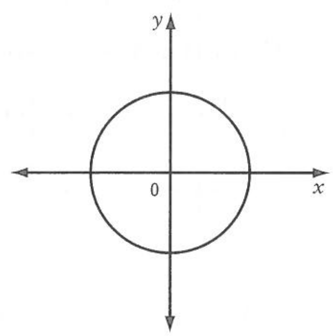
$\theta = \tan^{-1}(3)$

$= 1.25$

$\therefore$ The solution is 1.25 radians.

Correct solution:

7. a) Solve $\cos^2 \theta = \cos \theta$, $0 \leq \theta < 2\pi$. Give exact answers.



b) Suppose the domain of solutions to $\cos^2 \theta = \cos \theta$ is unrestricted. Write a general expression representing each solution. Can the expressions be combined to a single general expression representing all solutions? Why or why not?

Also try #13 on page 212 of *Pre-Calculus 12*.

8. Verify whether the expression $\frac{(1 + 2n)\pi}{3}$, $n \in I$ represents the general form for solutions to $\sec \theta = 2$.

Try various cases, such as $n = 0$.

Connect

9. Fill in the table with information related to solving trigonometric equations.

				Diagram
If	sin θ csc θ	< 0	the solutions will be in quadrants ________ and ________.	
If	sin θ csc θ	> 0	the solutions will be in quadrants ________ and ________.	
If	cos θ ________ θ	< 0	the solutions will be in quadrants ________ and ________.	
If	cos θ ________ θ	> 0	the solutions will be in quadrants ________ and ________.	
If	tan θ ________ θ	> 0	the solutions will be in quadrants ________ and ________.	
If	tan θ ________ θ	< 0	the solutions will be in quadrants ________ and ________.	