5.2 Transformations of Sinusoidal Functions

KEY IDEAS

- You can apply the same transformation rules to sinusoidal functions of the form $y = a \sin b(\theta - c) + d$ or $y = a \cos b(\theta - c) + d$.
  - A vertical stretch by a factor of $|a|$ changes the amplitude to $|a|$.

    $y = a \sin \theta$ $\qquad\qquad$ $y = a \cos \theta$

    If $a < 0$, the function is reflected through the horizontal mid-line of the function.
  - A horizontal stretch by a factor of $\frac{1}{|b|}$ changes the period to $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$ radians.

    $y = \sin (b\theta)$ $\qquad\qquad$ $y = \cos (b\theta)$

    If $b < 0$, the function is reflected in the y-axis.
  - For sinusoidal functions, a horizontal translation is called the *phase shift*.

    $y = \sin (\theta - c)$ $\qquad\qquad$ $y = \cos (\theta - c)$

    If $c > 0$, the function shifts c units to the right.
    If $c < 0$, the function shifts c units to the left.
  - The *vertical displacement* is a vertical translation.

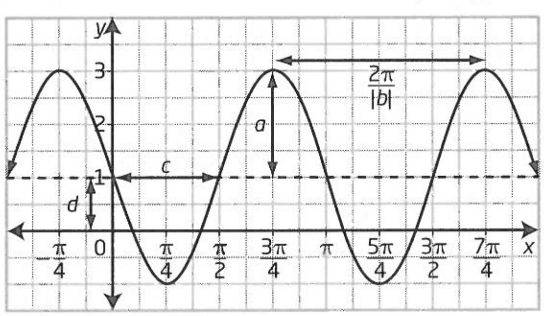
    $y = \sin \theta + d$ $\qquad\qquad$ $y = \cos \theta + d$
  - If $d > 0$, the function shifts d units up.
  - If $d < 0$, the function shifts d units down.

    $d = \frac{\text{maximum value} + \text{minimum value}}{2}$
  - The *sinusoidal axis* is defined by the line $y = d$. It represents the mid-line of the function.
- Apply transformations of sinusoidal functions in the same order as for any other functions:

  i) horizontal stretches and reflections, $\frac{1}{|b|}$
  ii) vertical stretches and reflections, $|a|$
  iii) translations, c and d
- The domain of a sinusoidal function is not affected by transformations.

  The range of a sinusoidal function, normally $\{y \mid -1 \le y \le 1, y \in R\}$, is affected by changes to the amplitude and vertical displacement.

  Consider the graph of $y = 2 \sin 2\left(x - \frac{\pi}{2}\right) + 1$.



$a = 2$, so the amplitude is 2

$b = 2$, so the period is $\frac{2\pi}{2}$, or π

$c = \frac{\pi}{2}$, so the graph is shifted $\frac{\pi}{2}$ units right

$d = 1$, so the graph is shifted 1 unit up

domain: $\{x \mid x \in R\}$

range: $\{y \mid -1 \le y \le 3, y \in R\}$

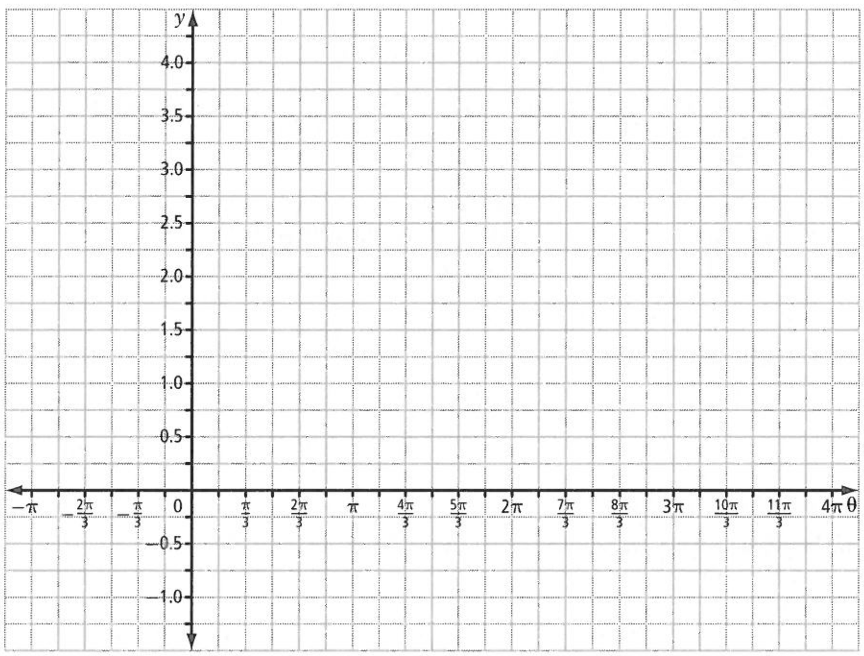
Working Example 1: Graph $y = \sin(\theta - c) + d$

a) Sketch the graph of the function $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$.

b) State the domain and range.

Solution

a) Sketch two cycles of the graph of the base function, $y = \sin \theta$.



Next, consider the transformed function $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$.

The amplitude is ________ and the period is ________.

$c =$ ________. This represents a phase shift of ________ units to the ________________.
(*left* or *right*)

$d =$ ________. This represents a vertical displacement of ________ units ________________.
(*up* or *down*)

On the grid above, sketch the sinusoidal axis at $y = 3$.

Use the sinusoidal axis, amplitude, period, and phase shift to sketch the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ on the grid above.

b) The domain of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$

is ____.

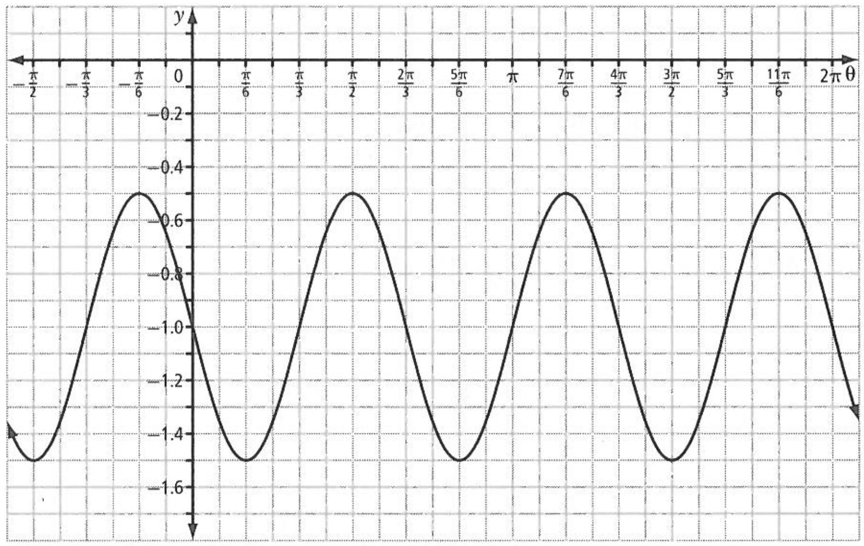
The range of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ is

____.

> Compare the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ to the graph of $y = \cos \theta$. What do you notice?

Working Example 2: Determine an Equation From a Graph

Write two sinusoidal equations of the form $y = a \sin b(\theta - c) + d$ and $y = a \cos b(\theta - c) + d$ to represent the function shown in the graph below.



Solution

Determine the equation of a sine function.
In $y = a \sin b(\theta - c) + d$, there are four parameters to determine: a, b, c, d.

- Determine the amplitude, a.

$$|a| = \frac{\text{maximum value} - \boxed{}}{2}$$

$$= \frac{\boxed{} - \boxed{}}{2}$$

$$= \_\_\_\_\_\_\_$$

- Determine the displacement, d.

  The equation of the sinusoidal axis is $y =$ _______. Therefore, $d =$ _______.

- Determine the value of b.

  The period of the graphed function is _______ radians.

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\_\_\_\_\_\_\_ = \frac{2\pi}{|b|}$$

$$b = \_\_\_\_\_\_\_$$

Choose b to be positive.

• Determine the phase shift, c.

The graph is shifted ________ units to the right, so $c =$ ________.

A sine equation that represents the graphed function is $y =$ ______________________________.

To determine an equation of a cosine function, $y = a \cos b(\theta - c) + d$, the values for a, b, and d are the same, but the phase shift is different.

The graph is shifted ________ units to the right, so $c =$ ________.

A cosine equation that represents the graphed function is $y =$ ______________________________.

Check Your Understanding

Practise

1. Determine the phase shift and the vertical displacement. Then, graph the function. Choose appropriate scales for the axes.

a) $y = \cos\left(\theta - \frac{\pi}{3}\right) - 1$

Phase shift: ________________ Vertical displacement: ________________

b) $y = \sin\left(\theta + \frac{\pi}{4}\right) + 2$

For more practice, see #1 and #2 on page 250 of *Pre-Calculus 12*.

2. Determine the key features of each sine function.

a) $y = -5 \sin\left(\frac{1}{2}(\theta - 90°)\right) + 15$

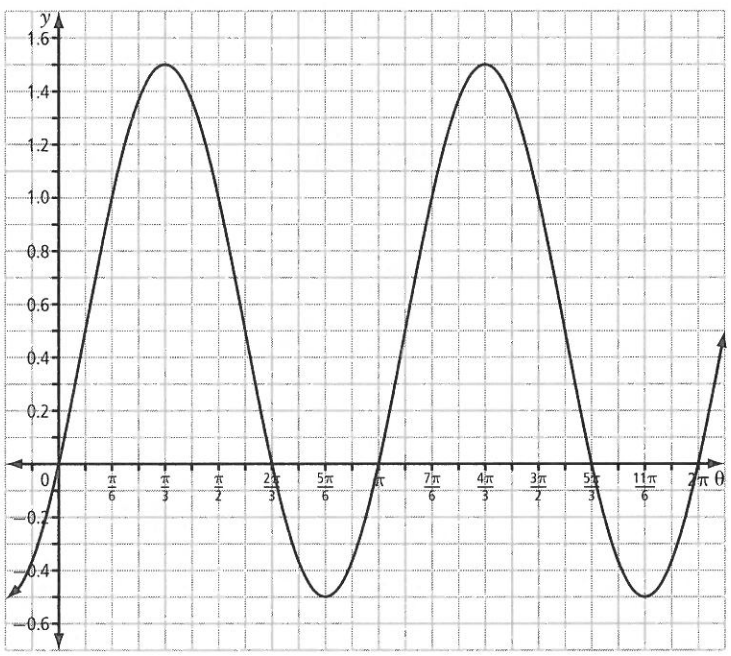
Amplitude: ______________________ Period: ______________________

Phase shift: ______________________ Vertical displacement: ______________________

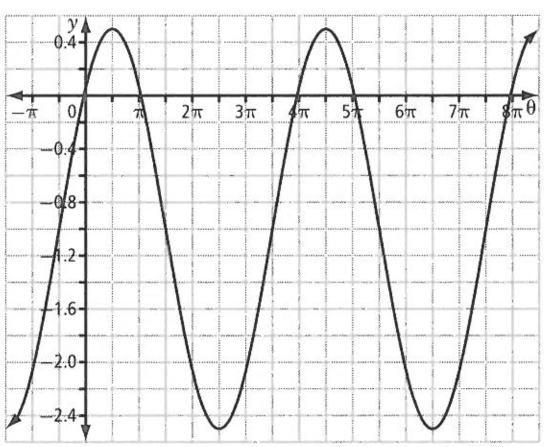
Domain: ______________________ Range: ______________________

b) $y = 0.1 \sin (2\theta + 90°) - 1$

c)



d)



3. Write the equation of each sine function in the form $y = a \sin b(x - c) + d$ given its characteristics.

   a) amplitude 2, period π, phase shift $\frac{\pi}{3}$ to the left, vertical displacement 1 unit down

   b) amplitude $\frac{1}{4}$, period 6π, phase shift π to the left, vertical displacement 2 units up

   c) amplitude 4, period 540°, phase shift 60° to the right, no vertical displacement

4. Graph each function in the space provided. Show at least two cycles.

   a) $y = 5 \sin 0.5(\theta + \pi) + 3$

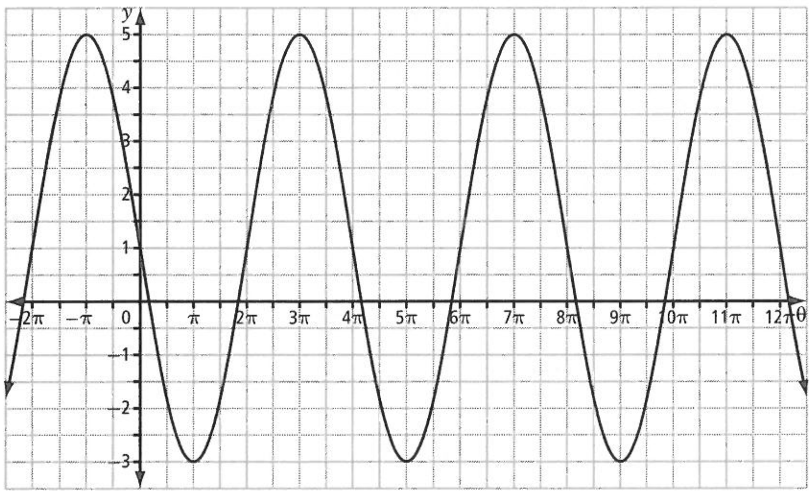
   b) $y = -2 \sin 2\left(\theta - \frac{\pi}{3}\right) + 4$

c) $y = 1.5 \cos 3\left(\theta + \frac{\pi}{2}\right) - 1$

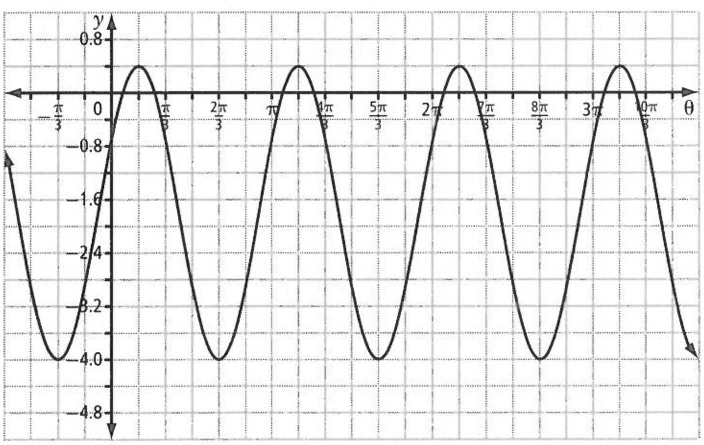
d) $y = -\cos \frac{1}{3}(\theta - \pi) + 3$

Apply

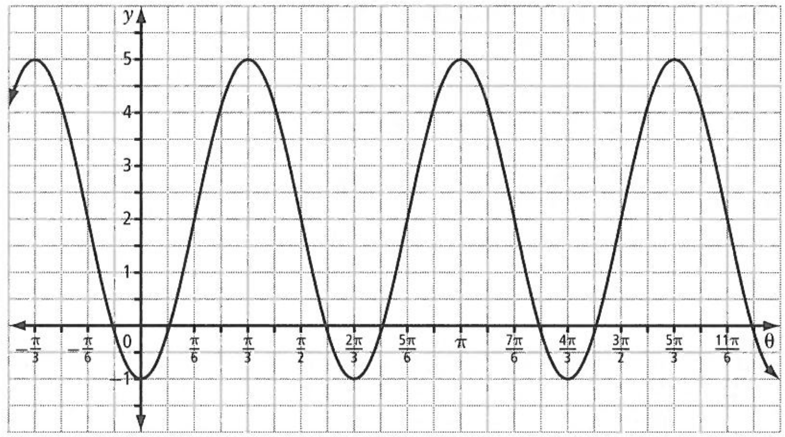
5. Write two different equations of the form $y = a \sin b(\theta - c) + d$ for the function graphed below. Use technology to check that your equations are correct.



6. Write two different equations of the form $y = a \cos b(\theta - c) + d$ to represent the function graphed below. Use technology to verify that your equations are correct.



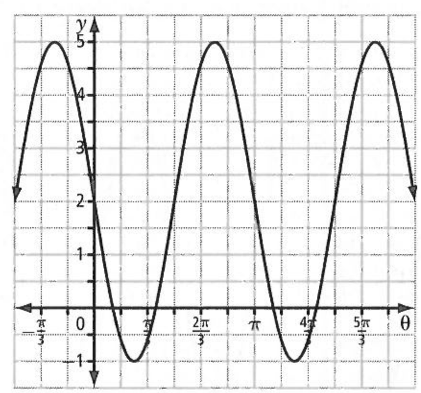
7. Write an equation of the form $y = a \sin b(\theta - c) + d$ and an equation of the form $y = a \cos b(\theta - c) + d$ to represent the function graphed below.



Connect

8. The graphed function is represented by an equation of each of the following forms. Determine the values of a, b, c, and d.

   a) $y = a \sin b(\theta - c) + d;\ a > 0$



   b) $y = a \sin b(\theta - c) + d;\ a < 0$

   c) $y = a \cos b(\theta - c) + d;\ a > 0$