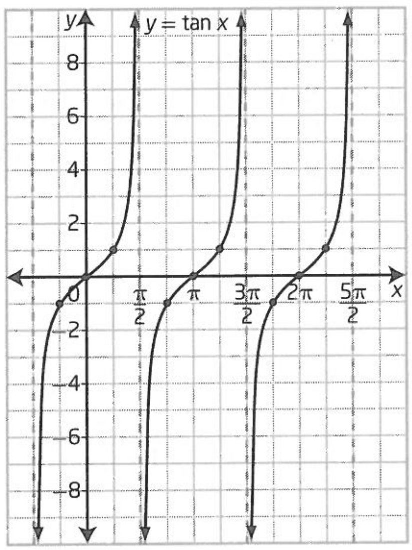
5.3 The Tangent Function

KEY IDEAS

- The graph of the tangent function, $y = \tan x$, is periodic, but it is *not* sinusoidal.



- These are the characteristics of the tangent function graph, $y = \tan x$:
  - It has period π or 180°.
  - It is discontinuous where $\tan x$ is undefined, that is, when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{\pi}{2} + n\pi$, $n \in I$. The discontinuity is represented on the graph of $y = \tan x$ as *vertical asymptotes*.
  - The domain is $\left(x \mid x \neq \frac{\pi}{2} + n\pi, x \in R, n \in I\right)$.
  - It has no maximum or minimum values.
  - The range is $\{y \mid y \in R\}$.
  - It has x-intercepts at every multiple of π: $0, \pi, 2\pi, \ldots, n\pi, n \in I$. Each of the x-intercepts is a turning point, where the slope changes from decreasing to increasing.

- On the unit circle, you can express the coordinates of the point P on the terminal arm of angle θ as (x, y) or $(\sin\theta, \cos\theta)$. The slope of the terminal arm is represented by the tangent function:

$$\begin{aligned}\text{slope} &= \frac{\Delta y}{\Delta x}\\ &= \frac{y-0}{x-0}\\ &= \frac{y}{x}\\ &= \tan\theta\end{aligned}$$

OR

$$\begin{aligned}\text{slope} &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\end{aligned}$$

Therefore, you can use the tangent function to model the slope of a line from a fixed point to a moving object as the object moves through a range of angles.

Working Example 1: Graph $y = \tan \theta$ Using Key Points

Graph $y = \tan \theta$ over the domain $-\pi \leq \theta \leq 4\pi$.

Solution

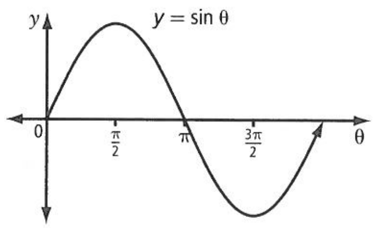
The key features needed to sketch the graph of the tangent function are the zeros, ones, and asymptotes.

> The *ones* of a tangent function are the values of θ when $y = \pm 1$.

Determine the Zeros

Given $\tan \theta = \frac{\sin \theta}{\cos \theta}$, when $\sin \theta = 0$, $\tan \theta =$ ________.

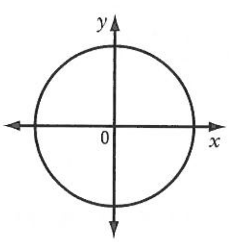
Therefore, $y = \tan \theta$ has the same zeros (x-intercepts) as $y = \sin \theta$.



The zeros of $y = \tan \theta$ over the domain $-\pi \leq \theta \leq 4\pi$ are ________, ________, ________, ________, ________, and ________.

Determine the Ones

The tangent function represents the slope of the terminal arm of an angle in standard position.



Given slope $= \frac{\sin \theta}{\cos \theta}$, the slope is 1 when $\sin \theta =$ ________.

Over the domain $-\pi \leq \theta \leq 4\pi$, the slope of the terminal arm is 1 when $\theta =$ ________, ________, ________, ________, and ________.

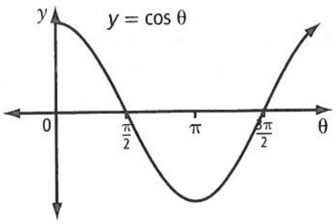
Given slope $= \frac{\sin \theta}{\cos \theta}$, the slope is -1 when $\sin \theta =$ ________.

Over the domain $-\pi \leq \theta \leq 4\pi$, the slope of the terminal arm is -1 when $\theta =$ ________, ________, ________, ________, and ________.

Determine the Asymptotes

Given $\tan \theta = \frac{\sin \theta}{\cos \theta}$, tan θ is undefined when ________ = 0.

Therefore, $y = \tan \theta$ has non-permissible values wherever $y =$ ________ has zeros (x-intercepts).

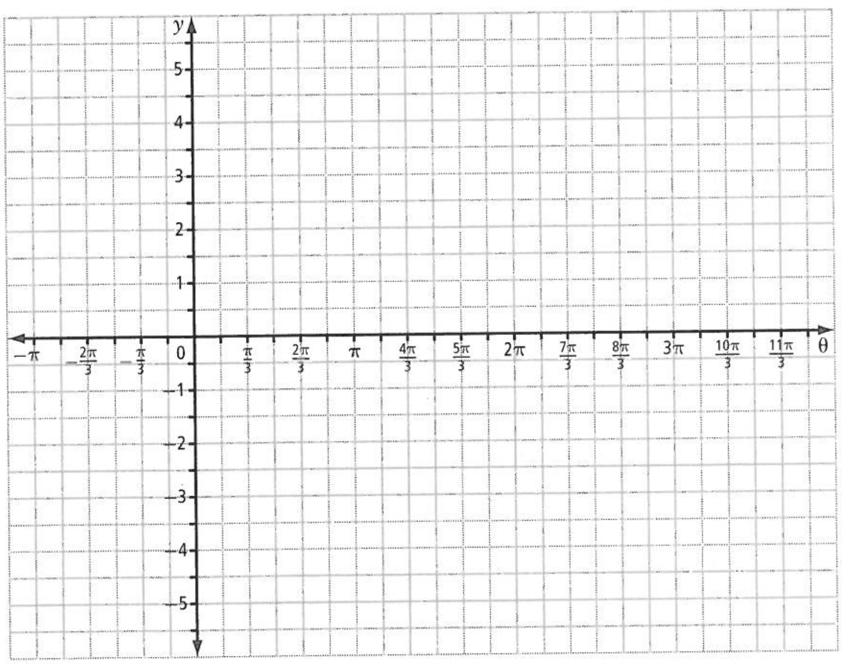


The non-permissible values of $y = \tan \theta$ over the domain $-\pi \leq \theta \leq 4\pi$ are ________, ________, ________, ________, and ________.

Use a broken line to draw a vertical asymptote on the graph at each non-permissible value of $y = \tan \theta$.

Plot the zeros and the ones.

Draw the tangent function starting at the lower edge of your graph near an asymptote, passing through the points plotted, and continuing to the upper edge of your graph near an asymptote. Be sure not to cross the asymptotes. In the given domain, how many cycles are shown?

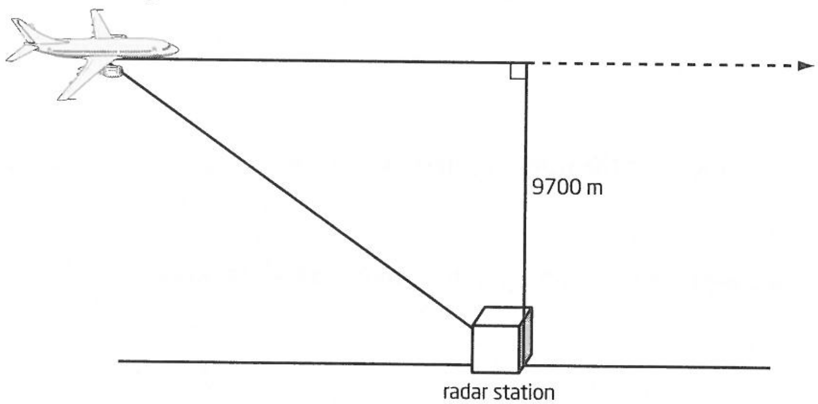


Working Example 2: Model a Problem Using the Tangent Function

An airplane is flying at a constant altitude of 9700 m in a straight line directly above a radar station. How does the horizontal distance between the plane and the radar station change as the plane crosses overhead?

Solution

Draw a diagram to illustrate the situation.



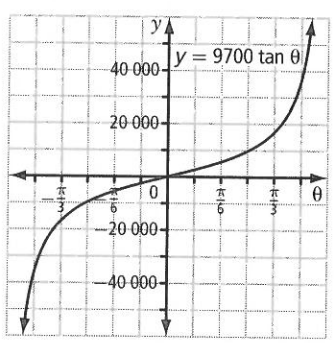
Label the complementary angle to the angle of elevation as θ, and label the horizontal distance as h.

Write an equation expressing h in terms of θ.

$\tan \theta =$

$h =$

As θ approaches 0, what happens to the horizontal distance?
Indicate this point on the graph of $y = 9700 \tan \theta$ below, and on the diagram above.



As the plane continues its flight away from the radar station,

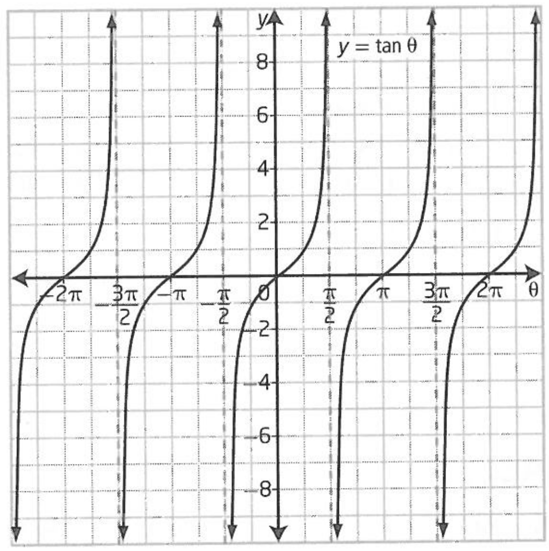
- what happens to θ?

- what happens to h? Indicate this region on the graph and on the diagram.

Check Your Understanding

Practise

1. Use the graph of $y = \tan \theta$ to determine each value.



a) $\tan 2\pi$

b) $\tan \frac{3\pi}{2}$

c) $\tan \frac{\pi}{4}$

d) $\tan \left(-\frac{\pi}{4}\right)$

2. Use the graph of $y = \tan \theta$ from #1 and your knowledge of the properties of the tangent function to determine each value.

a) $\tan (-\pi)$

b) $\tan (-3\pi)$

c) $\tan (-100\pi)$

3. Use the graph of $y = \tan \theta$ from #1 and your knowledge of the properties of the tangent function to determine each value.

a) $\tan \left(\frac{9\pi}{4}\right)$

b) $\tan \left(\frac{13\pi}{4}\right)$

c) $\tan \left(\frac{17\pi}{4}\right)$

4. Refer to your answers in #2 and #3. Write a general expression for all solutions in each case.

a) $\tan \theta = 0$

b) $\tan \theta = 1$

5. Use graphing technology to graph $y = \tan x$, where x is measured in degrees. Trace along the graph to locate the approximate value of the function when $x = 35°$. Predict the following values. Verify your predictions by tracing along the graph.

$\tan 35° \approx$ ________

a) $\tan (-325°)$

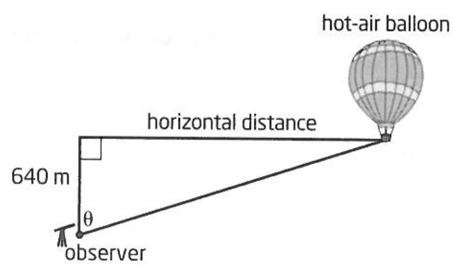
b) $\tan 395°$

c) $\tan (-35°)$

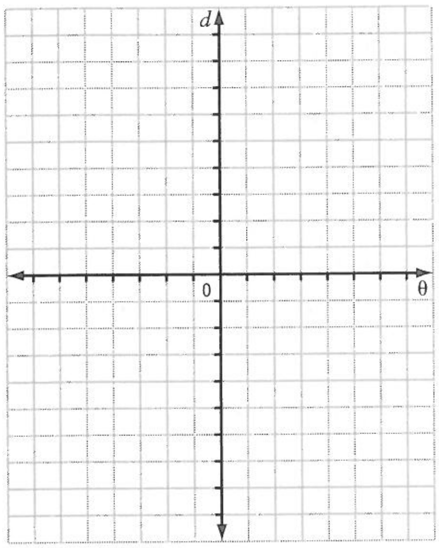
Apply

6. An observer watches a hot-air balloon pass directly overhead at a constant altitude of 640 m.

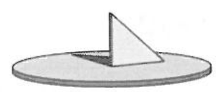
a) Determine the relation between the horizontal distance, *d*, in metres, from the observer to the hot-air balloon and the angle, in degrees, formed from the vertical to the balloon.



b) Graph the function. What are reasonable limits on the domain and range?



7. A simple sundial has a gnomon 10 cm high. Suppose the angle of the sun with respect to the gnomon changes from −75° at 7:00 a.m. to +75° at 5:00 p.m.



a) Write an expression for the length of the shadow in terms of the angle of the sun.

b) Describe any assumptions you made in creating this model.

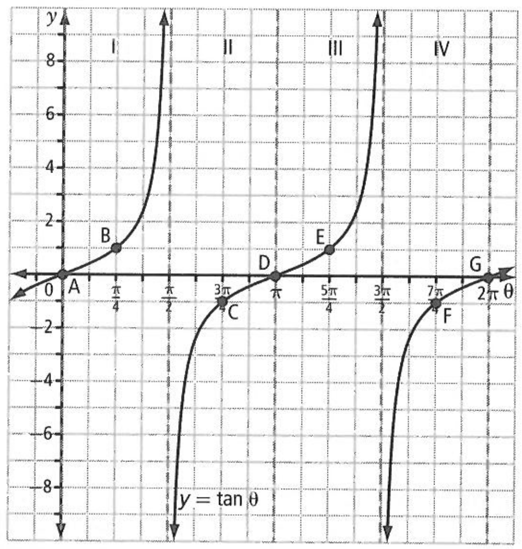
c) Determine the length of the shadow at 9:00 a.m.

d) Determine the length of the shadow at 4:00 p.m.

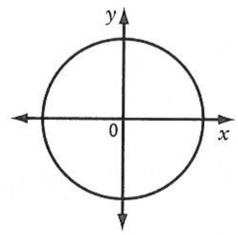
For more practice modelling using the tangent function, try #9 to #12 on pages 264–265 of *Pre-Calculus 12*.

Connect

8. a) For each labelled point on the graph of $y = \tan \theta$, sketch the terminal arm of angle θ on a unit circle.



A



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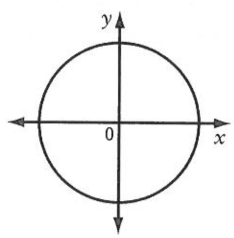

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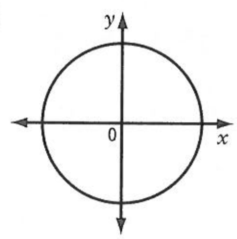

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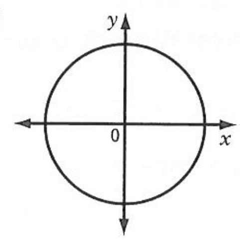

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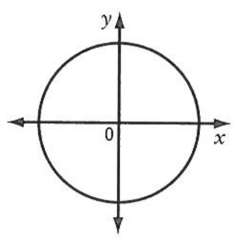

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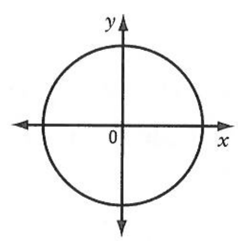

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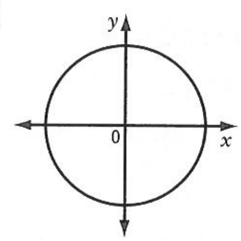
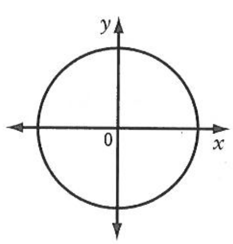


b) For each asymptote of $y = \tan \theta$, sketch the terminal arm of angle θ on a unit circle.

i) $\theta = \frac{\pi}{2}$



ii) $\theta = \frac{3\pi}{2}$