Composing Functions

Given two functions f and g and the given mapping

$$f: X \longrightarrow Y \qquad\qquad g: Y \longrightarrow Z$$

We understand that $x \mapsto y$ under the relation f and that $y \mapsto z$ under the relation g. Since g will act on elements that look like y and we can construct y by giving f elements that look like x. This gives us our composition of functions where f and g work together.

$$f: X \longrightarrow Y$$
$$g: Y \longrightarrow Z$$

$$f: x \mapsto y = f(x)$$
$$g: y \mapsto z = g(y)$$

And we can se that since $g(y) = z$ and $y = f(x)$ that $z = g\big(f(x)\big)$, which can be notated using circle notation as

$$z = (g \circ f)(x)$$

Example: Given some basic functions $f = \{(5,2), (7,-3), (4,8), (-2,-2)|$ and $g = \{(-3,4), (-2,5), (2,7), (8,2)\}$ determine the value of $f(g(-2))$ and $(g \circ g \circ f)(4)$

Solution: Understand that -2 is mapped under g and then that value is mapped under f

$$-2 \xrightarrow{\text{under } g} 5 \xrightarrow{\text{under } f} 2$$

So $f\big(g(-2)\big) = 2$. Likewise, we see that $(g \circ g \circ f)(4)$ will map 4 under f and then that value will be mapped under g and mapped under g again.

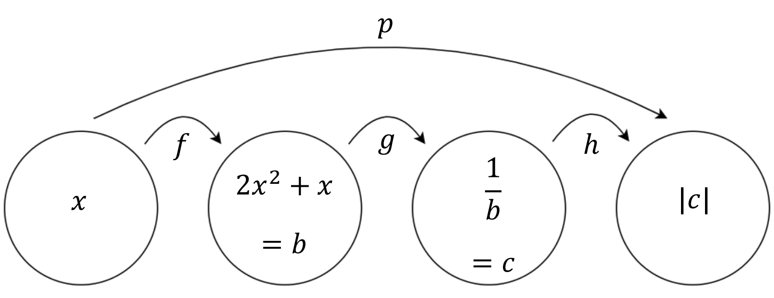
$$4 \xrightarrow{\text{under } f} 8 \xrightarrow{\text{under } g} 2 \xrightarrow{\text{under } g} 7$$

So $(g \circ g \circ f)(4) = g\Big(g\big(f(4)\big)\Big) = 7$

Example: If $f(a) = 2a^2 + a$ and $g(b) = \frac{1}{b}$ and $h(c) = |c|$. Determine the following:

$$p(x) = h\Big(g\big(f(x)\big)\Big) \qquad\qquad q(x) = (g \circ f \circ h)(x)$$

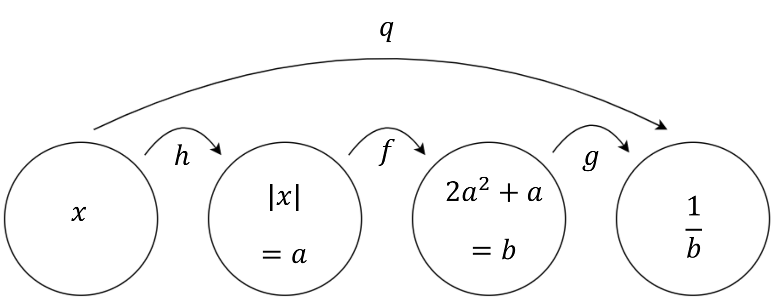
Solution: Start on the inside and work your way out.



And all we need to do is substitute c and b:

$$p(x) = |c| = \left|\frac{1}{b}\right| = \left|\frac{1}{(2x^2+x)}\right|$$

And it's the same idea for q, just a different order.



And we substitute b and a (Note that $|x|^2 = x^2$ so I don't bother writing the absolute value there)

$$q(x) = \frac{1}{b} = \frac{1}{2a^2+a} = \frac{1}{2x^2+|x|}$$

Just as we can compose functions together, it is important to be able to split functions apart by recognizing when a function (think operation) is inside another function (operation). Obvious operators are things such as:

$$\sqrt{(\ldots)},\quad \frac{1}{(\ldots)},\quad (\ldots)^n,\quad |(\ldots)|$$

But you should also be on the lookout for the same thing appearing in multiple places.

Example: If $f\left(g\big(h(x)\big)\right) = \left(\sqrt{x}-2\right)^2$ then determine f, g, and h.

Solution: Look at what happens to x.

1. Square root it
2. Subtract it by 2
3. Square it

First square root our variable will be h, so $h(x) = \sqrt{x}$. Then subtract 2 $g(x) = x - 2$. Finally, take the square of it, so $f(x) = x^2$

Things we need to know and understand:

- Functions are just actions that transform a number into a new number.
- A composition of functions is the natural way we stick functions (operations, actions) together.

Review Questions:

1. Explain how $f(x) = 2x + 1$ is a composition of functions.
2. If $f = \{(-2,4), (-5,-2), (-4,0), (2,2)\}$ and $g = \{(4,-4), (2,-5), (0,4), (3,5)\}$ then determine the following:
   a. $(g \circ f)(-2)$
   b. $f\left(f\big(g(2)\big)\right)$
   c. $f\left(g\left(f\big(g(4)\big)\right)\right)$
   d. Solve for x if $g\big(f(x)\big) = -4$
   e. $f \circ g$
   f. $g \circ f$
3. If $f(x) = 3\,x$ and $g(y) = \frac{1}{y}$ and $h(z) = 2 + \sqrt{z}$ then determine the following:
   a. $f\left(g\big(h(9)\big)\right)$
   b. $(g \circ h \circ f)(12)$
   c. $h\left(f\big(g(x)\big)\right)$
4. If $f(x) = \left(\sqrt{\left|5 + \frac{1}{2x}\right|}\right)$ the write $f(x)$ as a composition of:
   a. 2 functions
   b. 3 functions
   c. 4 functions
   d. 5 functions

Solutions:

1. This is a composition because there are two operations occurring. The first is times by 2, call this $g(x) = 2x$. The second operation is plus 1, call this $h(x) = x + 1$. The function f is the combination of action g and then action h.

$$f(x) = h\big(g(x)\big)$$

2.
   a. -4
   b. 4

c. Undefined
d. $x = -2$
e. $\{(4,0), (2,-2)\}$
f. $\{(-2,-4), (-4,4), (2,-5)\}$

3.
a. 0.6
b. 0.125
c. $2 + \sqrt{\frac{3}{x}}$

4. Answers will vary
a. Let $p(x) = 5 + \frac{1}{2x}$ and $q(x) = \sqrt{|x|}$. Then $f(x) = q\big(p(x)\big)$
b. Let $p(x) = 5 + \frac{1}{2x}$ and $q(x) = |x|$ and $r(x) = \sqrt{x}$. Then $f(x) = r\Big(q\big(p(x)\big)\Big)$
c. Let $p(x) = \frac{1}{2x}$ and $q(x) = 5 + x$ and $r(x) = |x|$ and $s(x) = \sqrt{x}$. Then $f = s \circ r \circ q \circ p$
d. Let $p(x) = 2x$ and $q(x) = \frac{1}{x}$ and $r(x) = 5 + x$ and $s(x) = |x|$ and $t(x) = \sqrt{x}$. Then

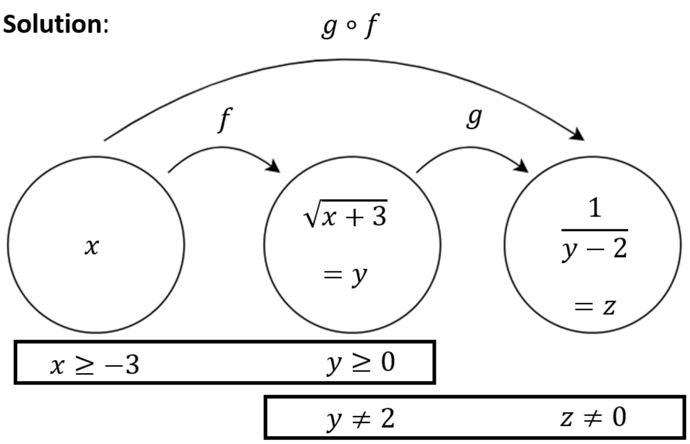
$$f = t \circ s \circ r \circ q \circ p$$

Analyzing Domain and Range of Function Compositions

To analyze the domain and range of a composition of functions we need to understand that the range of one function needs to match with the domain of the function that it is going to be composed with.

Example: If $f(x) = \sqrt{x+3}$ and $g(y) = \frac{1}{y-2}$, then determine the domain and range of $g \circ f$.

Solution:



The intersection of the range of f and domain of g is $[0,2) \cup (2,\infty)$ or alternatively written $[0,\infty) \setminus \{2\}$.
We want to see what happens if g acts on this set so take any $x \in [0,2) \cup (2,\infty)$. Then

$$0 \le x < 2 \quad \text{or} \quad 2 < x$$
$$-2 \le x - 2 < 0 \quad \text{or} \quad 0 < x - 2$$
$$\frac{1}{x-2} \le -\frac{1}{2} \quad \text{or} \quad 0 < \frac{1}{x-2}$$

$$\Rightarrow g(x) > 0 \;\text{ or }\; g(x) \le -\frac{1}{2}$$

The range is $(-\infty,-0.5] \cup (0,\infty) = \left\{y \mid y > 0 \text{ or } y \le -\frac{1}{2}\right\}$

We also want f to map onto the intersection so $f(x) \in [0,2) \cup (2,\infty)$ which just means don't have 2 as an output since $f(x) \in [0,\infty)$ regularly.

$$f(x) \ne 2 \Rightarrow \sqrt{x+3} \ne 2 \;\Rightarrow x \ne 1$$

The domain is $[-3,1) \cup (1,\infty) = \{x \mid x \ge -3,\; x \ne 1\}$

Things we need to know and understand:
- Functions are maps from one set (domain) to another set (range). The inverse is just the reverse mapping.
- The overall domain and range will be affected by the domain and range of the functions that make up the composition function.
- If you give a function a set, A, which is more than what it can accept then you need to adjust the domain of the function that is providing A.

- If you give a function a set, A, which is less than what it can usually accept then you need to adjust the range of function of what comes out after using A.

Review Questions:

5. If $f = \{(2,-2),(1,0),(-3,-3),(-5,0),(-4,-3)\}$ and $g = \{(0,-4),(2,-5),(-1,-4),(-3,4)\}$ then determine the domain and range of the following:
   a. $f(g(x))$
   b. $g \circ f$
   c. $g \circ f^{-1}$
   d. $g^{-1} \circ g$
6. If $f(x) = x^2 + 3$ and $g(x) = -|x| + 2$ then determine the domain and range of the following:
   a. $f \circ g$
   b. $g \circ f$
   c. $g \circ g$
7. If $f(x) = -\sqrt{x-1} - 2$ and $g(x) = 10 - x^2$ then determine the domain and range of the following:
   a. $g \circ f$
   b. $f \circ g$
   c. $f \circ f$
8. If $f(x) = |x| + 1$ and $g(x) = \frac{1}{x} - 1$ then determine the domain and range of the following:
   a. $f \circ g$
   b. $g \circ f$
   c. $g^{-1} \circ f$
9. If $f(x) = 2 - |x|$ and $g(x) = \sqrt{x-1}$ then determine the domain and range of the following:
   a. $f \circ g$
   b. $g \circ f$
   c. $f \circ g^{-1}$
   d. $g^{-1} \circ f$

Solutions:

5.
   a. $f \circ g = \{(0,-3),(2,0),(-1,-3)\}$ so $x = -1, 0,$ or 2 and $f(g(x)) = -3$ or 0
   b. $g \circ f = \{(1,-4),(-3,4),(-5,-4),(-4,4)\}$ so $x = -5,-4,-3,$ or 1 and $g(f(x)) = -4$ or 4
   c. $g \circ f^{-1} = \{(-2,-5),(-3,4)\}$ so $x = -3$ or -2 and $g(f^{-1}(x)) = -5$ or 4
   d. $g^{-1} \circ g = \{(0,0),(0,-1),(2,2),(-1,-1),(-1,0),(-3,-3)\}$ so $x = -3,-1,0$ or 2 and $g^{-1}(g(x)) = -3,-1,0$ or 2. Note that composition is not a function.
6.
   a. Domain: $\mathbb{R}$ and Range: $[3,\infty)$
   b. Domain: $\mathbb{R}$ and Range: $(-\infty,-1]$
   c. Domain: $\mathbb{R}$ and Range: $(-\infty,2]$
7.
   a. Domain: $[1,\infty)$ and Range: $(-\infty,6]$
   b. Domain: $[-3,3]$ and Range: $[-5,-2]$
   c. No valid domain. We say the domain is $\emptyset$ (the set of nothing). Hence, the range is also $\emptyset$
8.
   a. Domain $\mathbb{R} \setminus \{0\}$ and Range: $[1,\infty)$
   b. Domain $\mathbb{R}$ and Range: $(-1,0]$
   c. Domain $\mathbb{R}$ and Range: $(0,0.5]$
9.
   a. Domain: $[1,\infty)$ and Range: $(-\infty,2]$
   b. Domain: $[-1,1]$ and Range: $[0,1]$
   c. $g^{-1}(x) = x^2 + 1, x \geq 0$. With that in mind, Domain: $[0,\infty)$ and Range: $(-\infty,1]$
   d. Domain: $[-2,2]$and Range: $[1,5]$