Chapter 1 Review

The Concept of a Function

In the most basic form, functions are connections from set X (the domain) to set Y (the range). We called this connection a map and denoted the connection between sets (large X and Y) and elements inside the set (small x and y).

$$f: X \longrightarrow Y$$
$$f: x \mapsto y$$

We said that the output, y, could be denoted as $f(x)$ to indicate that it is the value of x after the function f operates on it.

This idea of input and output is crucial because functions will just take an input value (usually the dummy variable x) and *do stuff* to it in a certain order.

Example: $f(x) = \sqrt{x+5} - x$
This function takes the input and:

1. Adds 5 to it
2. Square roots it
3. Subtracts the original input from it

From these rules you can feed f anything and it will do those 3 things to it in that order.

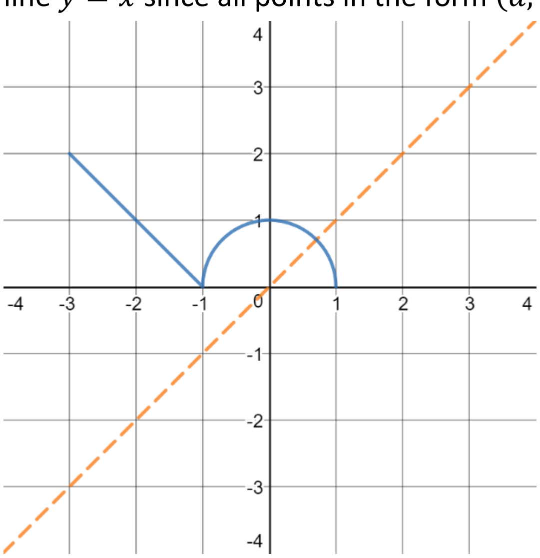
$$f: x+3 \mapsto x+8 \mapsto \sqrt{x+8} \mapsto \sqrt{x+8} - x - 3$$

So $f(x+3) = \sqrt{x+8} - x - 3$

> **Big Idea**: A well defined function f will perform a set list of actions on an input value x so the output value is $f(x)$. The input being called x is just to simplify the input. In reality the input can be anything. In this chapter we will be giving special attention to $f(ax+b)$ for horizontal transformations and $f(f^{-1}(\mathrm{x}))$ for checking inverses.

Inverses of Functions

When we invert a function, we go in the reverse order, that is from the range to the domain. If we have an ordered pair (x, y) it is implied that $x \mapsto y$ so the inverse would be $y \mapsto x$ and is the coordinate (y, x).
To graph an inverse, simply graph the new (y, x) coordinates. This amounts to reflecting the graph over the line $y = x$ since all points in the form (a, a) will stay the same and be invariant.



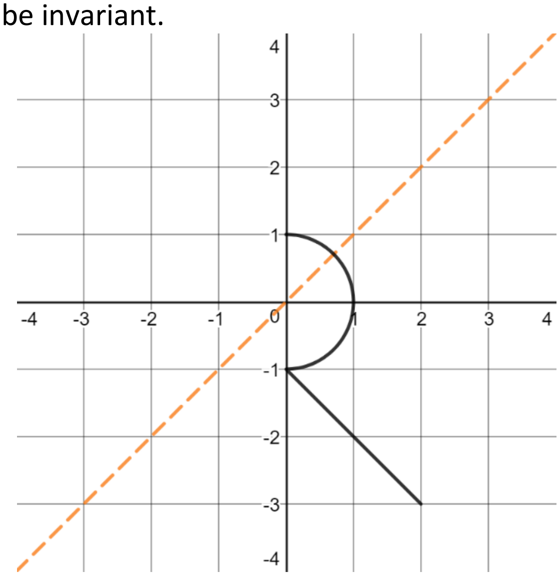
Reflect over the line $y = x$. Notice some key points that get mapped are:

$(-3,2) \mapsto (2,-3)$

$(-1,0) \mapsto (0,-1)$

$(0,1) \mapsto (1,0)$

$(1,0) \mapsto (0,1)$



Some functions, like the one above, are not 1-to-1 (some outputs can be mapped to from different inputs) and as a result their inverses are not well-defined (the same input will yield different outputs and it fails the vertical line test). Such functions do not have inverse functions (only *inverse relations*). To describe such an inverse as a function, the domain of the original function needs to be restricted so it is 1-to-1.

When finding the equation to an inverse, restrict the domain and then solve for x in $y = f(x)$. The function $x = f^{-1}(y)$ is the inverse and we simply swap the x and y labels so we can graph them on the same coordinate system.

Example: $f(x) = x^2 - 4$
Restrict the domain to be $x \leq 0$ so f is 1-to-1 on this interval. Then

$$\begin{aligned} y &= x^2 - 4 \\ x^2 &= y + 4 \\ x &= -\sqrt{y+4} \end{aligned}$$

Swap the labelling of variables

$$\Rightarrow f^{-1}(x) = -\sqrt{x+4}$$

We can check by confirming that $f\left(f^{-1}(x)\right) = x$

$$\begin{aligned} f\left(f^{-1}(x)\right) &= \left(-\sqrt{x+4}\right)^2 - 4 \\ &= x + 4 - 4 \\ &= x \end{aligned}$$

Big Idea: All 1-to-1 functions, $f: X \to Y$, have an inverse function, $f^{-1}: Y \to X$. Since the original function is just a series of instructions in order, the inverse is the same instructions in the reverse order and opposite operation so that everything is undone.

Transformations

To transform a function, f, we can stretch it or compress it in the vertical or horizontal directions, reflect it over either primary axis, and shift it in the vertical or horizontal directions.

In function notation these transformations look like:

$$a \cdot f\left(\frac{1}{b}(x-h)\right) + k$$

And the equivalent mapping notation looks like:

$$(x, y) \mapsto (bx + h, ay + k)$$

Where a, b are vertical and horizontal stretch factors respectively. If they are negative there is a reflection over an axis. The vertical and horizontal shift amounts are given by k and h. Written in this manner expansions/compressions/reflections must be done before translations since we follow order of operations in the mapping notation.

*** THIS DOES NOT MEAN IT IS IMPOSSIBLE TO SHIFT BEFORE WE STRETCH!! Just for this form where the function notation has the coefficient factored out of x as in $\frac{1}{b}(x-h)$

If you see $f(cx + d)$ you can interpret this as a horizontal shift d units and then a stretch by c units OR put it in the standard form $f\left(c\left(x + \frac{d}{c}\right)\right)$ as a horizontal stretch by c units and then a shift by d/c units.

Note that mapping notation does exactly what it says horizontally and vertically. If it multiplies by a number with absolute value larger than 1, then it will be an expansion as the values gets larger. If it multiplies it by a small number with absolute value less than 1, then it will be a compression as the value gets smaller. Positive shifts are right and up and negative shifts are left and down. Function notation reverses these ideas horizontally.

If $y = f\left(\frac{1}{b}(x-h)\right)$ then

$$f^{-1}(y) = \frac{1}{b}(x-h)$$
$$b \cdot f^{-1}(y) = x - h$$
$$b \cdot f^{-1}(y) + h = x$$

When we solve for x, we get that old x values, $f^{-1}(y)$, are being multiplied by b and added by h, the exact same as mapping notation!

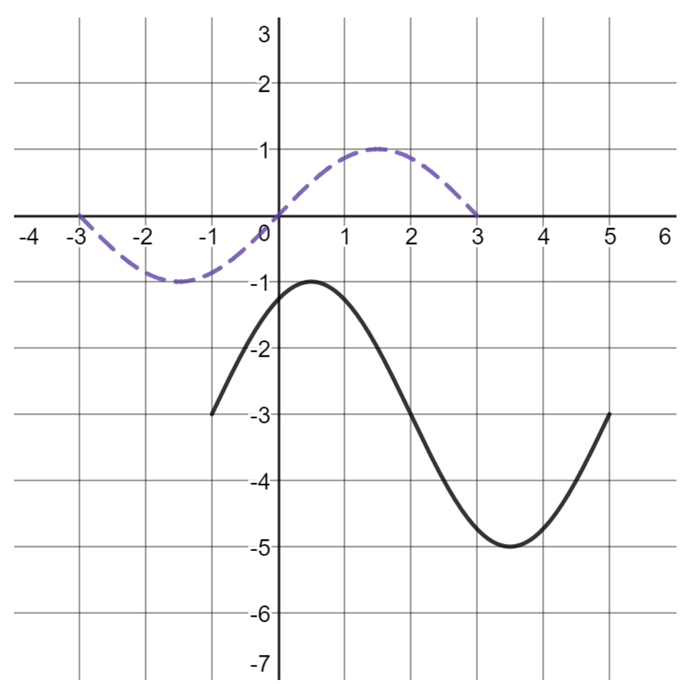
Since mapping notation does exactly what it says, my suggestion is to find the mapping notation first and then put it into function notation if necessary.

You are expected to take either a set of verbal instruction, a graph of a function and the image function, function notation, or mapping notation and create the other 3 from it. To find a graph of the image function, we can either do each transformation step by step, or use mapping notation to see where coordinates get mapped to.

Example: Graph $2f(-x+2)-3$
Interpret as a shift left 2 and then reflection over y-axis OR change to standard form with factored coefficient $2f\big(-(x-2)\big)-3$ and see it as a reflection over y-axis and then a shift right 2.
Change to mapping notation $(x,y) \mapsto (-x+2, 2y-3)$ OR $(x,y) \mapsto (-(x-2), 2y-3)$ depending on interpretation.
Use mapping notation to find the image coordinate of a few points:

$(-3,0) \mapsto (5,-3)$
$(-1.5,-1) \mapsto (3.5,-5)$
$(0,0) \mapsto (2,-3)$
$(3,0) \mapsto (-1,-3)$

Connect points with similar curves



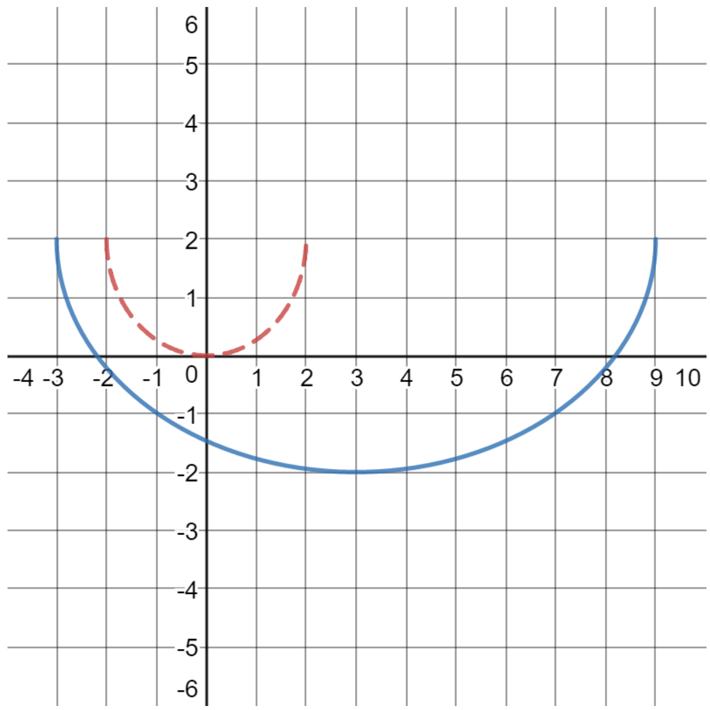
When finding the equation to a transformed graph. Use knowledge of how certain transformations change certain characteristics of the graph.

Example: When we stretch there can be invariant points such as intercepts and asymptotes. These key characteristics can be used to determine the shifts.

Example: Domain and range are will change under all operations, but the size will remain invariant under shifts and they can be used to determine the expansion and compression factors.

The domain tripled in length (horizontal expansion by 3) and the range doubled in length (vertical compression by 2). The origin was mapped to $(3, -2)$, so a shift right by 3 and down 2.

$$2f\left(\frac{1}{3}(x-3)\right)-2$$



There are a lot of characteristics you have learned about: Domain, range, zeros (including multiplicity), intercepts, end behaviour, and asymptotes. Think about how these characteristics change under certain transformations and remain unchanged under others.

Big Idea: Expanding and compressing functions multiplies either the x or y values by some number. Shifts will be obtained by adding to the x or y values. Mapping notation is exactly this and function notation will invert the horizontal operations. The order of transformations is important and dictated by the mapping notation order of operations. If multiplication is done first then we stretch first, otherwise if addition is done first we shift first.

REVIEW MATERIAL

Chapter Review: Page 56 – 57

Practice Test: Page 59 Short and Long Answer

Workbook Review and Sections: Class website under Additional Material