Chapter 2 Review

GEOMETRIC SEQUENCES AND SERIES

Geometric sequences are built when each term gets multiplied by a common ratio to build the next term

$$a_1, ra_1, r^2a_1, r^3a_1, \dots, r^na_1$$

$$\Rightarrow a_k = a_1 r^{k-1}$$

We get the property that

$$\frac{a_{k+1}}{a_k} = r$$

If we want to add up the first n terms of a geometric sequence, then we get the following:

$$S_n = a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1$$
$$rS_n = \qquad ra_1 + r^2a_1 + r^3a_1 + \cdots + r^na_1$$

And take the difference

$$(1-r)S_n = a_1 - r^na_1$$

$$\Longrightarrow S_n = \frac{a_1(1-r^n)}{1-r}$$

And finally, we looked at infinite geometric series as $n \to \infty$

$$S_\infty = \frac{a_1}{1-r}$$

Since $r^n \to 0$ when $|r| < 1$. Note that if $|r| \geq 1$ then the series will not converge and is not defined.

Big Ideas you need to know and understand:

- The summation of a series requires you to label a "first term", count how many terms are being added, and determine the common ratio.
- For infinite series, it converges when $|r| < 1$ because then the exponential $r^x \to 0$ as $x \to \infty$

Review questions:

Determine the sum of the following series

1. $S = 1 - 3 + 9 - 27 + \cdots + 6561$
2. The 4th term of the sequence is 10 and the 6th term is 22.5. What is the sum of the first 6 terms?
3. The number of new babies born into a population each year can be modelled by a geometric sequence. In the first year there are 40 babies, in the 3rd year there are 164 babies. How much has the population grown in the first 5 years?

Solutions:

1. $a_1 = 1,\ r = -3,\ n = 9 \Longrightarrow S_9 = \frac{1(1+3^9)}{1+3} = 4921$
2. $a_4 = 10,\ a_6 = 22.5 \Longrightarrow \frac{a_6}{a_4} = r^2 = 2.25 \Longrightarrow r = \pm 1.5$. You could find a_1 since $a_4 = a_1 r^3$ $\Longrightarrow a_1 = \pm 2.\overline{962}$. However, you could just look at the series in reverse where $b_1 = 22.5$ and $r = \pm\frac{2}{3}$.

In each case we get $S_6 = \frac{22.5\left(1-\left(\frac{2}{3}\right)^6\right)}{1\mp\frac{2}{3}} = 61.5\overline{740},\ 12.3\overline{148} = \frac{\pm 2.962\left(1-(1.5)^6\right)}{1\mp 1.5}$

3. We have $a_1 = 40$ and $a_3 = 164 = 40 \cdot r^3 \Longrightarrow r = 1.60$. We want to find S_5

$$S_5 = \frac{40(1-1.6^5)}{1-1.6} = 634 \text{ new animals}$$

SIGMA NOTATION

We can define a sum using sigma notation

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \cdots + f(b-1) + f(b)$$

Here, k is the index that starts from a and increases by 1 until it stops at b. We add each result to each other

This allows us to rewrite S_n in terms of sigma notation

$$S_n = a_1 + a_1 r + \cdots a_1 r^{n-1}$$

$$= \sum_{k=0}^{n-1} a_1 r^k$$

$$= \frac{a_1(1-r^n)}{1-r}$$

Big Ideas you need to know and understand:

- Sigma notation is just a condensed way to write summations
- The index k is a dummy variable and could be labeled anything

Review questions:

Evaluate the following:

4.
$$\sum_{k=1}^{10} \frac{(1.3)^k}{3}$$

5.
$$\sum_{k=-3}^{5} (-2)^k \cdot \frac{4}{5}$$

6.
$$\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k \cdot 8$$

Write the following using sigma notation

7. $3 + 6 + 12 + 24 + \cdots + 768$
8. $1215 - 405 + 135 - 45 + \cdots - \frac{5}{9}$

Solutions:

4. $a_1 = \frac{1.3}{3},\ r = 1.3,\ n = 10 \Longrightarrow S_{10} = \frac{\left(\frac{1.3}{3}\right)(1-1.3^{10})}{1-1.3} = 18.468 \ldots$
5. $a_1 = -\frac{1}{10},\ r = -2,\ n = 9$ (from -3 to 5 is nine terms) $\Longrightarrow S_9 = \frac{\left(-\frac{1}{10}\right)(1+2^9)}{1+2} = -17.1$
6. $a_1 = \frac{9}{2},\ r = \frac{3}{4},\ n = \infty \Longrightarrow S_\infty = \frac{4.5}{1-0.75} = 18$
7. $\sum_{k=0}^{8} 3 \cdot 2^k = \sum_{k=1}^{9} 3 \cdot 2^{k-1}$
8. $\sum_{k=0}^{7} 1215 \cdot \left(-\frac{1}{3}\right)^k = \sum_{k=1}^{8} 1215 \cdot \left(-\frac{1}{3}\right)^{k-1} = \sum_{k=-2}^{5} (-5) \cdot (-3)^k$