Chapter 3 Review

Dividing Polynomials

We discussed how to reduce the rational expression $\frac{p(x)}{x-a}$ into the form $q(x) + \frac{r}{x-a}$ where $q(x)$ is the quotient and r is the remainder. More generally we can take $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ where $\deg(r) < \deg(d)$.

While doing long division, it is important to remember that when you subtract the terms from the original polynomial you are just subtracting two polynomials.

Example: Divide $x^3 + 3x - 1$ by $x + 1$

Solution:

$$\begin{array}{r|l} & x^2 - x + 4 \\ x+1 & x^3 + 0x^2 + 3x - 1 \\ & -(x^3 + x^2) \\ \hline & \quad -x^2 + 3x - 1 \\ & \quad -(-x^2 - x) \\ \hline & \qquad 4x - 1 \\ & \qquad -(4x + 4) \\ \hline & \qquad r = -5 \end{array}$$

There's nothing wrong with subtracting term by term and including $0x^2$ as a term, and it can be helpful so that you don't subtract unlike terms.

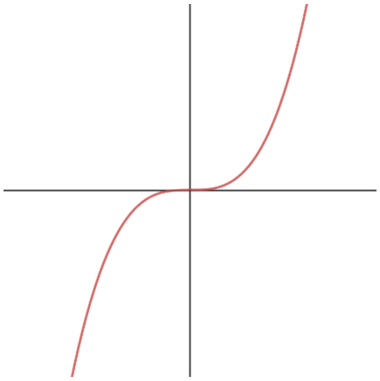
Use ***Integer Root Theorem*** to find possible factors of the polynomial $a_n x^n + \cdots + a_1 x + a_0$ any factor of the form $(x - A)$ must have the property that A is a factor of a_0. **Remember we are only factoring over $\mathbb{Z}[X]$.** Furthermore, we can use **Factor Theorem** to see if $(x - A)$ is a factor as $p(A) = 0$ if it is.

Use ***Remainder Theorem*** anytime we are talking about the remainder of a polynomial, $p(x)$, after it has been divided by a binomial $x - a$. The remainder is $p(a)$.

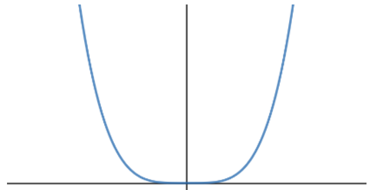
Characteristics of Polynomials

The Equation: A polynomial can be written as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for real a_k and whole number n.

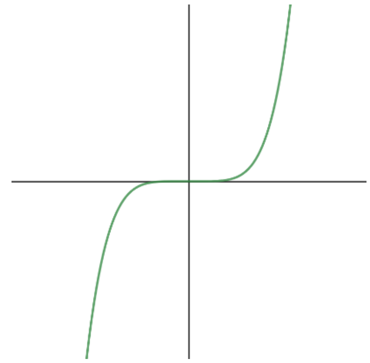
End Behaviour: What does the polynomial look like for large values of x (both positive and negative)? The leading term, $a_n x^n$, determines the behaviour and the sign of a_n will determine the direction.



x^3



x^4



x^5

Behaviour Near Zero: What does the polynomial look like for small values of x (both positive and negative)? The final terms, $a_2x^2 + a_1x + a_0$, determines what the polynomial looks like near $x = 0$.

Y-intercept: The y-intercept is when $x = 0$ and for our polynomial that value is the constant term a_0.

Zeros: To find where the polynomial crosses the x-axis we need to factor the polynomial using remainder and factor theorem and some division. A possible factoring of p could lead us to:

$$p(x) = a_n(x - r)(x - s)^2(x - t)^3$$

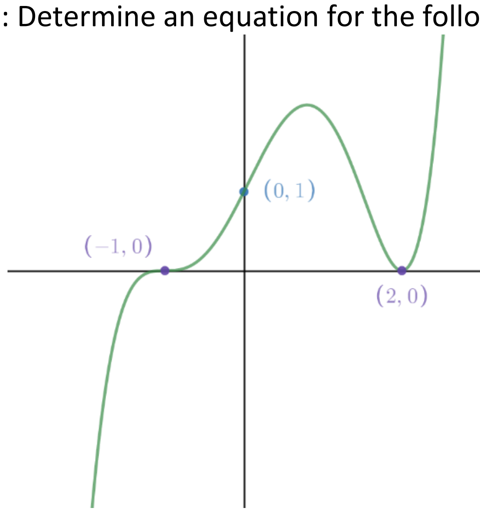
Where r has a ***multiplicity*** of 1 (passes straight through the axis), s has a multiplicity of 2 (does not pass through, but touches the axis), and t has a multiplicity of 3 (passes through the axis, while turning).

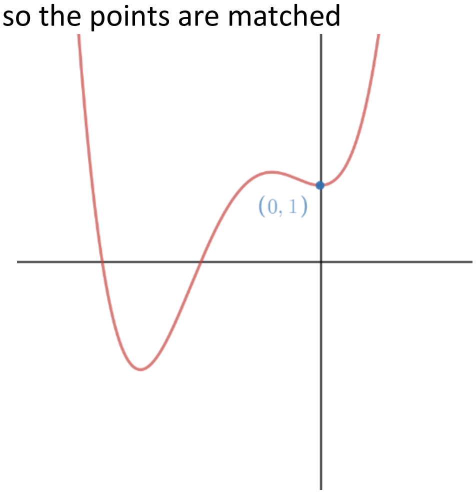
EQUATIONS OF POLYNOMIALS

To find the equation of a polynomial we have two possibilities:

1. *The multiplicity of the zeros adds to the degree of the polynomial.*
   In this case we can write the polynomial in factored form while matching multiplicities as powers. Keep in mind you will need to solve for the leading coefficient.
2. *The multiplicity of the zeros does NOT add to the degree of the polynomial.*
   You can guess the leading term and the constant term and use the behaviour near $x = 0$ to sort out the sign of some of the smaller terms.
   OR you can shift the function so you have enough zeros and then shift it back

Example: Determine an equation for the following polynomials so the points are matched





The zeros have multiplicity 3 and 2 which give a degree 5 polynomial. To match the y-inercept we will vertically stretch the function.

$$p(x) = a(x+1)^3(x-2)^2$$
$$p(0) = 1$$
$$\Rightarrow 1 = a \cdot 4 \Rightarrow a = 0.25$$

$$p(x) = 0.25(x+1)^3(x-2)^2$$

Here, we need to match the shape. On a large scale it looks like x^4, on a small scale near $x = 0$ it looks like a flat line so $y = 1$ and then a bit of an upward parabola and positive cubic.
Something like $y = x^4 + x^3 + x^2 + 1$ is a good start.

Using technology, we could make adjustments to the coefficients. but we probably want the x^4 coefficient to be small so we can see the other characteristics a bit better.

Alternatively, if we shift the function down 1 unit it has zeros at $x = 0$ (with a multiplicity of 2) and then around $x = -1$ and $x = -3$. With these we get

$$\begin{aligned} y &= (x+3)(x+1)x^2 + 1 \\ &= x^4 + 4x^3 + 3x^2 + 1 \end{aligned}$$

Things we need to know and understand:

- How to sketch a polynomial given the factored and expanded form.
- How to use factor theorem and remainder theorem to factor a polynomial.
- How to determine when a polynomial is positive or negative.
- How to divide polynomials and express them in quotient and remainder form.
- How remainder theorem relates the polynomial outputs.

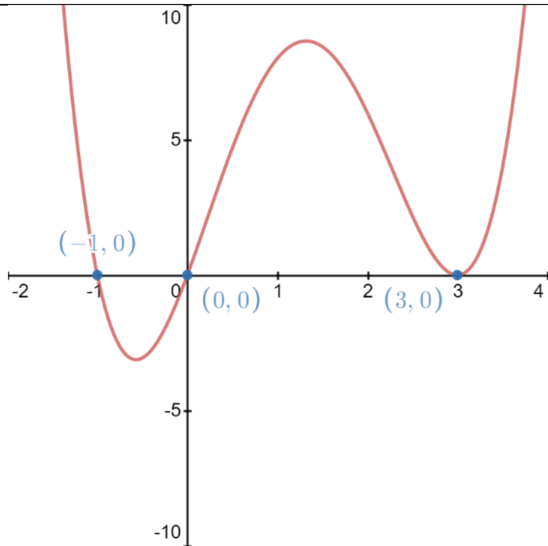
Review Questions:

1. Consider the polynomial $p(x) = x^5 - x^3 - 8x^2 + 8$.
   a. Factor $p(x)$
   b. Sketch it and state the intervals it is positive and negative.

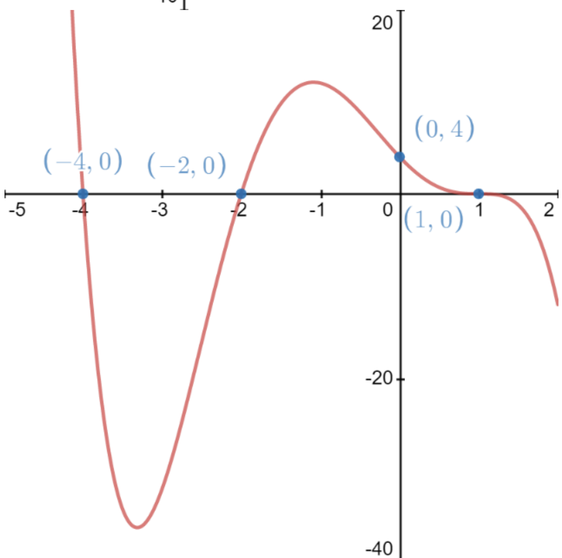
   Write it in the form $q(x) + \frac{r(x)}{d(x)}$ when you divide it by the polynomial $d(x)$ if:

   c. $d(x) = x + 2$
   d. $d(x) = x^2 + 1$
   e. $d(x) = x^4 + 1$

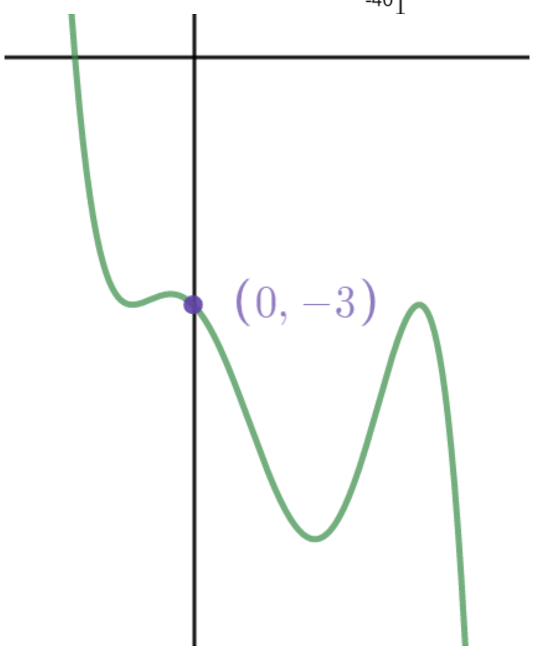
2. Given the following graphs determine an equation for a polynomial that passes through the indicated points.



a.



b.

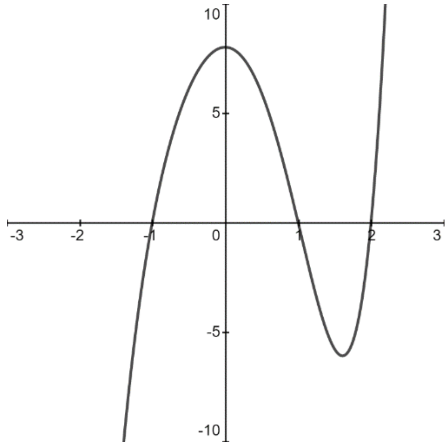


c.

3. Construct a quadratic and cubic polynomial that have remainders of 2 when divided by x, -1 when divided by $x-3$, and 1 when divided by $x+1$.

Solutions:

1.
   a. $(x-1)(x+1)(x-2)(x^2+2x+4)$. Note that x^2+2x+4 is prime in $\mathbb{Z}[X]$ as no two numbers multiply to 4 and add to 2.

   b. 

   $p(x) > 0$ if $-1 < x < 1$ or $x > 2$. It is negative if $x < -1$ or $1 < x < 2$

   c. $x^4 - 2x^3 + 3x^2 - 14x + 28 - \frac{48}{x+2}$

   d. $x^3 - 2x - 8 + \frac{2x+16}{x^2+1}$

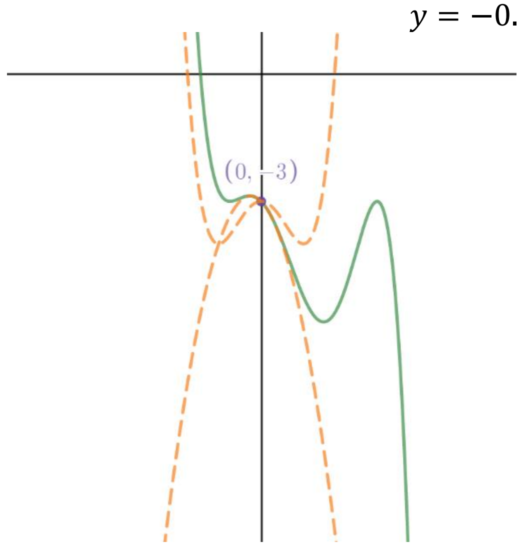
   e. $x - \frac{x^3+8x^2+x-8}{x^4+1}$

2.
   a. $(x+1)(x-3)^2x$

   b. $-0.5(x-1)^3(x+2)(x+4)$

   c. Something of the form $-x^5 + x^4 - x^2 - x - 3$, the exact equation used was

$$y = -0.25x^5 + x^4 - 2x^2 - x - 3$$



3. The quadratic has the form $q(x) = ax^2 + bx + c$ and we know that $q(0) = 2, q(-1) = 1$, and $q(3) = -1$. Therefore $c = 2$ and we have

$$1 = a - b + 2$$
$$-1 = 9a + 3b + 2$$

Solving for a and b we get $a = -\frac{1}{2}$ and $b = \frac{1}{2}$

The cubic can have the form $p(x) = Ax^3 + Bx + 2$. Solve for A and B as before and get

$$1 = -A - B + 2$$
$$-1 = 27A + 3b + 2$$

So that $A = -\frac{1}{4}$ and $B = \frac{5}{4}$