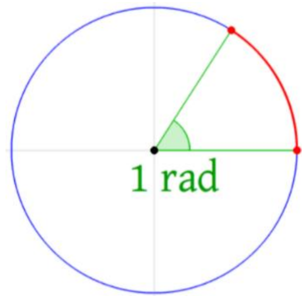
Chapter 4 Review

THE RADIAN

Rather than dividing the circle into 360 slices and calling each slice "1 degree", we define the angle based on the size of the arc length that is made.



1 Radian is the angle when the Arc Length is 1 Radius long.

In general, the angle in radians is defined as

$$\theta = \frac{\text{Arc Length}}{\text{radius}}$$

From this definition we can determine the angle all around the circle. Set the arc to be the circumference, $2\pi r$, and the radius to be r. Then

$$\theta = \frac{2\pi r}{r} = 2\pi \stackrel{\text{def}}{=} 360°$$

Rather than thinking of the quadrants using $0°,\ 90°,\ 180°,\ 270°,\ 360°$ as markers and then placing the angle in between these, you should be thinking of $0,\ 0.5\pi,\ \pi,\ 1.5\pi,\ 2\pi$ as markers.

Example: What quadrant is $\frac{8}{3}\pi$ in?

Solution: We see that the fraction is 8 thirds of π. This means we are taking π (half a rotation) and cutting it into thirds, then making 8 copies of that. 3 thirds on the top, 3 thirds on the bottom, and then 2 thirds more to make 8 puts it in quadrant 2.

Alternatively, we see the fraction $\frac{8}{3}$ as a decimal is about 2.7 which means it's 0.7 more than a full rotation. This is more than 0.5 but less than 1 so it's in quadrant 2.

The angle $\frac{8}{3}\pi$ means the arc length is $\frac{8}{3}\pi$ times as long as the radius.

Coterminal Angles are angles that will have the same terminal arm. Since a full rotation is 2π radians, all coterminal angles just multiples of a full rotation more or less than a given angle.

$$\theta + 2\pi n, n \in \mathbb{Z}$$

Big Ideas you need to know and understand:

How the radian is defined using the arc length.

A half rotation is $180° = \pi$ radians and a full rotation is $360° = 2\pi$ radians.

How to find coterminal angles in a restricted domain.

Review questions:

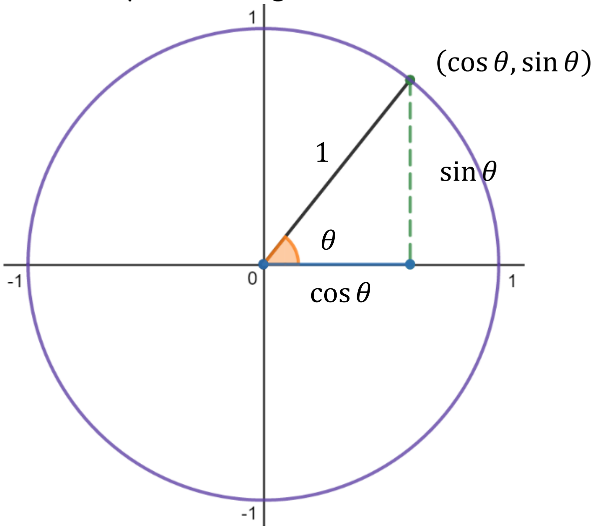
1. How is the radian defined?
2. A circle rotates at 25 rad/min, what is the speed of the wheel if it's radius is 20cm?
3. Find the general form of all angles coterminal to $\frac{11}{7}\pi$. Sketch the angle and find coterminal angles in $[-3\pi, \pi]$.

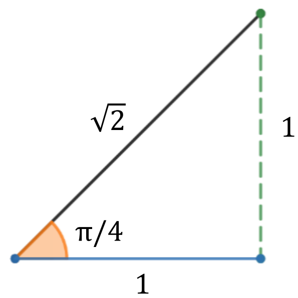
Solutions:

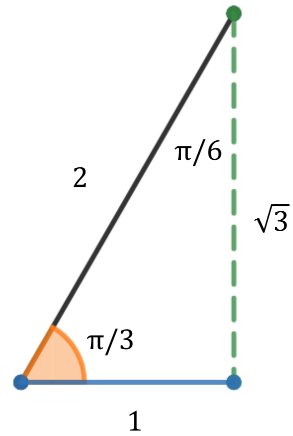
1. See above
2. 500cm/min or 0.3km/h
3. $\frac{11}{7}\pi + \frac{14}{7}\pi n, n \in \mathbb{Z}$. The angle is in Quad IV. Coterminal angles are $-\frac{3}{7}\pi$ and $-\frac{17}{7}\pi$

The Unit Circle and Special Triangles

The unit circle is used to show trig ratios of special angles, multiples of $\pi/4$ (this includes $\pi/2$) and $\pi/6$ (this includes $\pi/3$). On the unit circle the coordinates (x, y) are $(\cos\theta, \sin\theta)$. To find these exact coordinates we use two special triangles.







Angle, θ	$\sin\theta$	$\cos\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	Undefined
Reference Angle in Quadrant II	Positive	Negative	Negative
Reference Angle in Quadrant III	Negative	Negative	Positive
Reference Angle in Quadrant IV	Negative	Positive	Negative

Use the special triangle to find the ratio for your reference angle and then adjust the sign depending on the quadrant you are in. Remember sine is the y value and positive in quadrant 1 and 2 and cosine is the x value and positive in quadrant 1 and 4.
The unit circle is great to see the value of the trig ratios at multiples of $\pi/2$ where there is no triangle, but the coordinate is on the coordinate axis.

Big Ideas you need to know and understand:
Memorize the special triangles and the quadrants the trig ratios are positive in.
Sine and cosine are bound between -1 and 1.
Sine is associated with the y value and cosine is associated with the x value.
Tangent is defined as sine ÷ cosine.

Review questions:
4. For what angles on $[0,2\pi)$ are sine as cosine maximized and minimized?
5. Why doesn't it matter that the special triangles won't fit into the unit circle (their hypotenuses are not 1)?

Solutions:
4. Sine is maximized and minimized at $\pi/2$ and $3\pi/2$. Cosine is maximized and minimized at 0 and π.
5. The ratios stay the same no matter the size of the triangle as long as the angles stay the same. Since the ratios stay the same, we can find the correct coordinates on the unit circle even if the triangle I gave you does not fit exactly inside it.

SOLVING FOR THE RATIO

We introduced 3 new trig ratios that are defined as the **reciprocal** of the 3 basic trig functions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Note that these are the reciprocal and NOT the inverse. When finding the ratios, we can use the special triangles or unit circle of the reference angle and then adjust the sign for the quadrant.

Example: If $\theta = -\frac{4}{3}\pi$, then the reference angle is $\pi/3$ and in Quadrant 2. We can use the previous table to find the trig ratios and see that $\csc\left(-\frac{4\pi}{3}\right) = \frac{2}{\sqrt{3}}$; $\sec\left(-\frac{4\pi}{3}\right) = -2$; and $\cot\left(-\frac{4\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

If the angle is not an angle that is on a special triangle or unit circle, you can use your calculator to find the approximate trig ratio.

Example: If $\theta = 5$ then:
$\csc 5 = \frac{1}{\sin 5} = \frac{1}{-0.96} = -1.04$; $\sec 5 = \frac{1}{\cos 5} = \frac{1}{0.28} = 3.52$; and $\cot 5 = \frac{1}{\tan 5} = \frac{1}{-3.38} = -0.30$

If we are given 1 of the six trig ratios and can determine the quadrant the angle is in, we can determine the other 5 trig ratios by making an appropriate triangle with 2 of the 3 sides and the final side using Pythagoras.

Example: If $\sin\theta = \frac{2}{3}$ and $\theta \in (\pi/2, \pi)$. Then the opposite side is 2 and the hypotenuse is 3 and we are in quadrant 2, therefore the adjacent leg is $\sqrt{5}$. So $\cos\theta = -\sqrt{5}/3$, $\tan\theta = -2/\sqrt{5}$, $\csc\theta = 3/2$, $\sec\theta = -3/\sqrt{5}$, and $cot\,\theta = -2/\sqrt{5}$.

Big Ideas you need to know and understand:
Memorize the 3 reciprocal trig ratios.
Understands that trig functions ARE functions that take an angle and map it to the associated ratio of side lengths.

Review questions:

6. If $\theta = 17\pi/4$ determine the 3 reciprocal trig ratios.
7. If $\theta = 6\pi/5$ determine the 3 reciprocal trig ratios.
8. Determine the 3 reciprocal trig ratios if $cos\,\theta = 0.35$ and the angle is in Quadrant 4.
9. Why is the domain of cosine all real numbers?
10. What is the domain of secant not all real numbers?

Solutions:

6. $\csc\theta = \sec\theta = \sqrt{2}, \cot\theta = 1$
7. $\csc\theta = -1.701, \sec\theta = -1.236, \cot\theta = 1.376$
8. $\csc\theta = -1.068, \sec\theta = 2.857, \cot\theta = -0.374$
9. Any angle will make a terminal arm that intersects the unit circle. The x coordinate of the intersection will be the value of cosine.
10. The domain of secant is all real numbers except values of the form $\pi/2 + \pi n, n \in \mathbb{Z}$. This is where cosine is 0 and secant has 0 as a denominator and hence a vertical asymptote.

SOLVING FOR THE ANGLE

When we solve for the angle, we have to use the **inverse** of the trig function. This takes the ratio and maps it to an associated angle that would give that ratio. Note that this is just *an* angle and not *the* angle since the trig functions are not one-to-one. We need to restrict the domain of the angle in order to justify talking about the inverse trig function. The angle of sine are restricted to be in the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the angles of cosine are restricted to be in the domain $[0, \pi]$. The angles of tangent are restricted to be in the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ This is what your calculator will give you as a solution. You need to use this angle to find the correct reference angle and the actual solution you want.

Example: Find the general solution to $\sin\theta = -0.6$. Then $\theta = \sin^{-1}(-0.6) = -0.644$, this is the calculator answer. The general solution is all angles coterminal to this so $-0.644 + 2\pi n, n \in \mathbb{Z}$. Additionally, there is a second solution in Quadrant 3. The angle will have a reference angle of the same magnitude and so the second general solution is $-\pi + 0.644 + 2\pi n = -2.498 + 2\pi n, n \in \mathbb{Z}$.

Finding solutions in a specific domain basically amounts to finding this general solution and then finding the correct coterminal angles as you did in the very first section of this chapter. Cosine is even so the solutions will be the positive and negative and tangent has a period of π so not much more work needs to be done to find other solutions.

We can solve for angles in trig functions that are in a polynomial.

Example: Find the general solution of

$$2\cos^2(3x) + \cos(3x) - 1 = 0$$

This is a quadratic. Replace $\cos(3x)$ with y. Then solve $2y^2 + y - 1 = 0$. This can be factored OR use the quadratic equation, $(2y - 1)(y + 1) = 0 \implies y = \frac{1}{2}$ or $y = -1$. Once the quadratic has been solved, we can go back to the trig function and finish solving for x.

$$\cos(3x) = \frac{1}{2}, \text{ we can use special triangles to see } 3x = \pm\frac{\pi}{3} + 2\pi n \text{ and so } x = \pm\frac{\pi}{9} + \frac{2\pi n}{3}, n \in \mathbb{Z}$$

Big Ideas you need to know and understand:
Understands that inverse trig functions ARE functions that take a specific ratio and map it to the associated angle. In order to be a function, we need a restricted domain of the forward function.
How to interpret a calculator answer to find the general solution or solution in a certain domain.
Recognize ratios of special triangles and the appropriate angle it belongs to.

Review questions:

11. What is the domain of $\sin^{-1} x$?
12. What is the exact value of $\sin(\tan^{-1}(1/4))$?
13. What are the solutions to $2\sec(x) + 3 = 0$ on the interval $[-\pi, \pi]$?
14. What are the solutions to $\csc^2\big(2(x+1)\big) + \csc\big(2(x+1)\big) = 6$ on the interval $(0,2)$?
15. When is $\cos\left(\frac{\pi}{5}x\right)$ maximized on the interval $(0,10]$?

Solutions:

11. $[-1,1]$, the possible y values on the unit circle.
12. $1/\sqrt{17}$ (if $\theta = \tan^{-1}\left(\frac{1}{4}\right)$ then the opposite side is 1 and the hypotenuse is $\sqrt{17}$)
13. $\pm$2.301
14. 0.309, 0.741, 1.972, and 2.403
15. 10