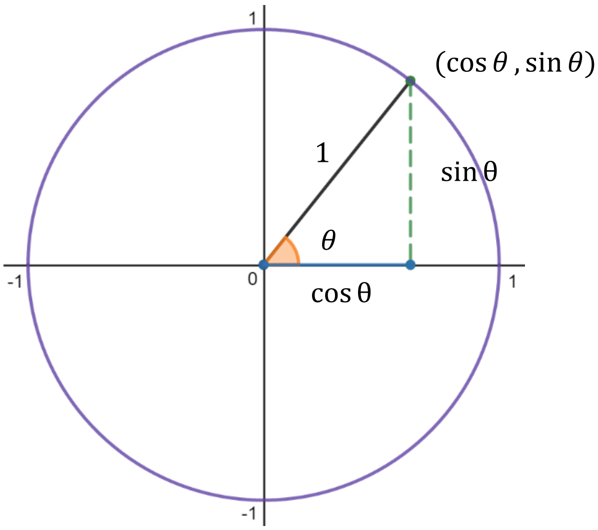
Chapter 5 Review

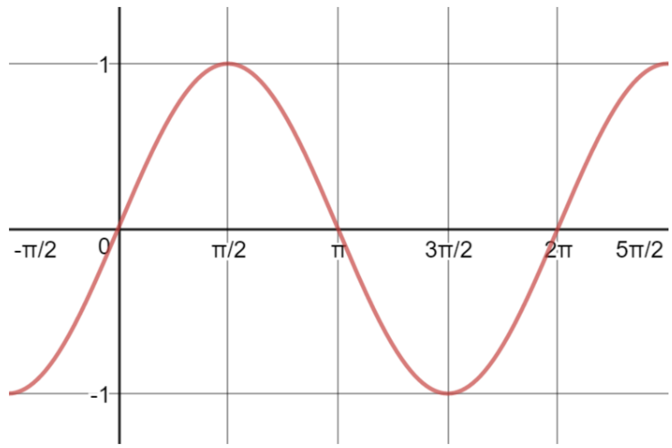
The Sine and Cosine Graph

We use the unit circle to graph sine and cosine. Sine is the y value and will start at 0 and make the following path:

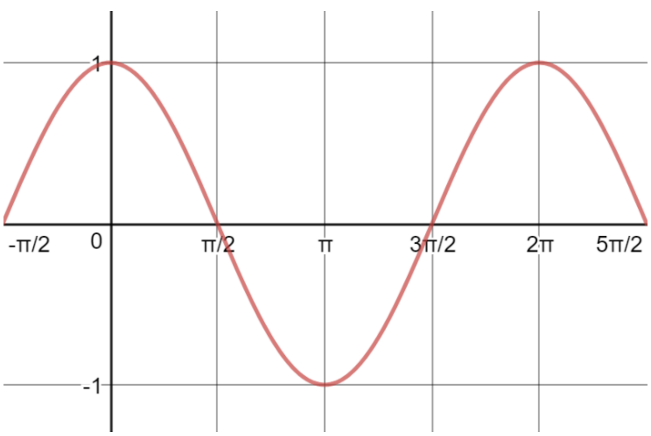
$$0 \to 1 \to 0 \to -1 \to 0$$

Which results in the graph





Cosine, on the other hand start at 1 and make the following path $1 \to 0 \to -1 \to 0 \to 1$



We can see that cosine is just a shift of sine, so their characteristics are virtually the same, just that starting point. The **amplitude** is half of the vertical distance from maximum to minimum. On sine and cosine, the amplitude is 1. The **period** is the horizontal distance it takes to repeat the pattern (from max to max, or min to min). Both sine and cosine have a period of 2π. The **midline** is the horizontal line that cuts the graph in half. Sine and cosine have a midline at $y = 0$ (the x-axis).

There is *a lot* of symmetry in the graph of sine and cosine! Use the symmetry to help you make nice graphs!

Big Ideas you need to know and understand:
Memorize the shape of sine and cosine. That means knowing key points it passes through (zeros and extremas) and the basic characteristics of the graph (period, amplitude, midline, domain, range, etc.)
Cosine is just sine shifted by 90° or $\pi/2$.

Review questions:

1. Graph a full period of sine and cosine. Label key points.
2. Why do cosine and sine make the same shape?

Solutions:
1. See above graphs.
2. Both graphs are looking at the value of the x OR y coordinates on the unit circle. Since the unit circle is symmetric over each axis, the set of x points is the same as the set of y points they just have a different order.

Transforming the Sine/Cosine Graph

We want to be able to graph and find the equation to trig functions of the form

$$a \cdot \sin\big(b(x-c)\big) + d$$

$|a|$: The **amplitude** of the trig function. If $a < 0$ then the graph will move in the opposite direction (ie sine goes down first, cosine goes up first).
b : The change in **period.** The new period is $2\pi/b$
c : The **phase shift**. How far left or right the graph is shifted.
d : The **vertical displacement**. How far up or down the midline was moved.

If we are given a graph or a set of points and asked to find the equation to it, we should start by finding the midline, then amplitude, then the period of our function. LEAVE THE PHASE SHIFT FOR LAST!
** If we know the period, T, then we have $b = 2\pi/T$

Once we know the above, we can decide if we have a sine or cosine graph and then measure what the phase shift should be.

Example: Find two equations to the function that has a maximum at $(0,3)$ and nearest minimum is $(-7,-5)$

Solution: It might help to sketch it out as you go along. The midline is -1 as that is halfway between 3 and -5. The amplitude is 4 since it is four units from the midline to the max. The period is 14 since it takes half a period to go from -7 to 0.
Our equations should look like

$$\pm 4\sin\left(\frac{\pi}{7}(x-c)\right) - 1$$
$$\pm 4\cos\left(\frac{\pi}{7}(x-c)\right) - 1$$

To fill in the sign of the function and the value of c we need to understand the shape of our function. Cosine starts at the top, same as our function (no phase shift and positive), sine starts at the middle (-3.5 or 3.5 for our graph) and goes up and then down.

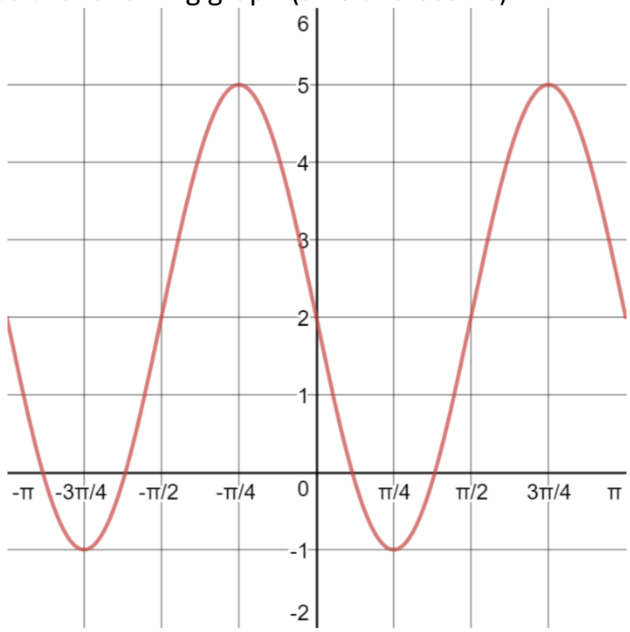
$$-4\sin\left(\frac{\pi}{7}(x-3.5)\right) - 1$$
$$4\cos\left(\frac{\pi}{7}x\right) - 1$$

Graphing these functions is a bit easier since all the needed information is right in front of you on your equation.

Big Ideas you need to know and understand:
Graphing complex trig functions is just applying transformations of the basic sine and cosine graph.
There is no one equation that will match the graph of a trig function. Since they are periodic there will be infinitely many equations that work plus sine and cosine varieties.

Review questions:

3. What is the equation to the following graph (sine and cosine)?



4. What is the equation (sine and cosine) to a trig function that has an amplitude of 2 and passes through the midline at the points $(4, -3)$ and $(10, -3)$ (consecutive midline points)?
5. Graph the function $1.5\sin\big(\pi/4(\theta - 3)\big) + 0.5$

Solutions:

3. $-3\sin(2x) + 2$ OR $3\cos\big(2(x + \pi/4)\big) + 2$
4. $\pm 2\sin\left(\frac{\pi}{6}(x + 2)\right) - 3$ OR $\pm 2\cos\left(\frac{\pi}{6}(x - 1)\right) - 3$ (plus minus because we don't know if it starts going up or going down first.)
5. Use Desmos or Geogebra to check.

SOLVING BY GRAPHING AND BUILDING EQUATIONS

When solving equations graphically, we are looking for approximate solutions rather than exact solutions. While we can solve some equations algebraically, it is not always possible, and we may not need a very accurate result.

When we are building an equation to model something periodic, we work to abstract the problem to find its basic characteristics.

- The period is the length of 1 full cycle. Typically, the period is a measurement of time (things change periodically over time), but it doesn't have to be (maybe period is a measure of distance or something ese).
- The amplitude will be half of the max minus the min, just like with points. It will be measured in units of your output (what your equation is modelling, ie height, temperature, etc).
- The vertical displacement is the average value of the output (what your equation is modelling, ie height, temperature, etc).

- To determine the phase shift, you need to consider the initial conditions given to you in the problem. Sometimes there are no initial conditions and you can choose no phase shift, but often you will need to match the constraints of the problem.

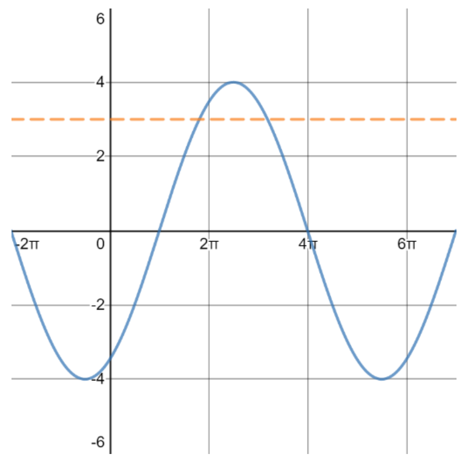
Big Ideas you need to know and understand:
Solving by graphing is fine in most instances to get an approximate solution. We can use algebra to solve these equations sometimes; however, this is not always possible, nor is it required in most everyday problems.
Building equations from word problems requires the same steps as the previous section.

Review questions:

6. What is the general solution to the following equation:
$$4\sin\left(\frac{1}{3}(x-\pi)\right)=3$$
7. A Ferris wheel has a radius of 15m and picks up passengers 3m from the ground and then moves up first. The wheel makes one full rotation every 90 seconds. If the highest point passengers reach is 32m above the ground after 40 seconds. Determine an equation for the height of the passenger at time t. What times in the first 2 minutes will the passenger be 10m above the ground?

Solutions:

6. The graph should look like:



My quick guess is that $x \approx 5, 10 + 6\pi n, n \in \mathbb{Z}$

Solving it exact we get
$$\frac{x-\pi}{3}=0.8, \qquad 2.3+2\pi n$$
$$x = 5.685\ldots, \qquad 10.022\ldots + 6\pi n$$

7. $h(t) = 15\cos\left(\frac{2\pi}{90}(t-40)\right)+17$

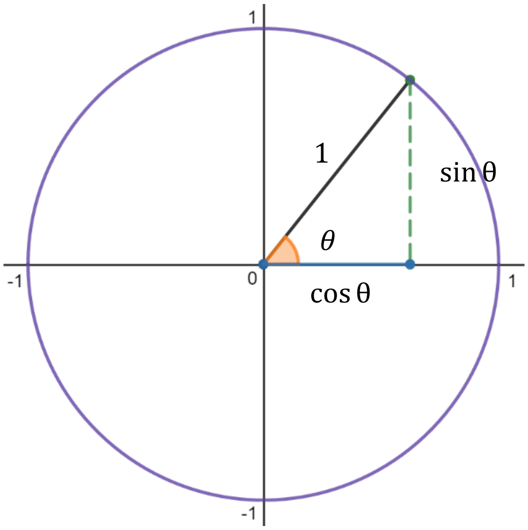
If $h(t)=10$, then $\frac{2\pi}{90}(t-40) = -2, 2 + 2\pi n, n \in \mathbb{Z}$. Solving for t we get
$$t = 11, 69 + 90n$$
So when $t \le 120$ we get $t = 11, 69, 101$ seconds

Chapter 6 Review

THE PYTHAGOREAN IDENTITY

We use the unit circle and a right-angle triangle to prove our first identity. On the unit circle we can form a right-angle triangle with hypotenuse length 1 and height $sin\,\theta$ and base $cos\,\theta$. Pythagoras' Theorem states that the sum of the squares of the two smaller sides will be the square of the hypotenuse. Therefore,

$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$



In addition to this identity, we can construct 2 other ones by dividing by sine or cosine

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow 1 + \cot^2\theta = \csc^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

Big Ideas you need to know and understand:
Understand how to build the Pythagorean identities from a right-angle triangle just as outlined above and in class.
Recognize that when you see a trig term to a power greater than 1 and/or the number 1 in a statement that this identity will be helpful.

Review questions: Prove the following statements are true for all permissible values.

8. $\frac{2\sin^2 x + 2\cos^2 x}{\csc x} = 2\sin x$
9. $\sqrt{\frac{1-\cos^2 x}{\cos^2 x}} = |\tan x|$
10. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

Solutions:

8. We factor the numerator to $2(\sin^2 x + \cos^2 x) = 2(1) = 2$. Then $2/\csc x = 2 \div (1/\sin x) = 2\sin x$
9. The numerator is equivalent to $\sin^2 x$, so inside the radical is $\sin^2 x/\cos^2 x = \tan^2 x$. We know that $\sqrt{x^2} = |x|$ which completed the proof
10. Difference of squares so $\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = (1)(\sin^2 x - \cos^2 x)$. We can then replace $\cos^2 x = 1 - \sin^2 x$ and we have finished

ADDING AND SUBTRACTING ANGLE IDENTITIES

We built the adding and subtracting angle identities using geometry. If you want to see the proof of that you can look back at our notes, but it is not expected of you to derive the identities.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

When using the identities be careful that cosine's identity does not have the same sign on both sides of the identity (a sum of angles is a difference of terms and vice versa).

One important thing we can do with these identities is find the exact value for new angles that are of the form

$$\frac{n\pi}{12}, n \in \mathbb{Z}$$

Example: $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Additionally, from these two identities we can build the special case where the angles are the same, $\alpha = \beta$. In that case we get that $\sin(2\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$, hence

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

And we also get that $\cos(2\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$ so that

$$\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

This last identity has multiple forms because of the Pythagorean trig identity which allows us to express $\sin^2 x$ in terms of $\cos^2 x$.

Big Ideas you need to know and understand:
Sine and cosine addition/subtraction identities are useful when finding the sine or cosine of a new angle $(A + B)$ is hard but finding the sine and cosine of the parts (A and B) is easy.
You need to know that the double angle identities are just special cases of the general additive and subtractive identities when the angles are the same.

Review questions:

11. If angles A and B are in quadrants I and II respectively, with $\sin A = \frac{3}{7}$ and $\sin B = \frac{4}{7}$ then determine $\sin(A + B)$ and $\cos(A + B)$.
12. Write as a single sinusoidal expression:
$$4\sin x - 2\cos x$$
13. Find the exact value of $\tan(\pi/12)$
14. Prove the following is true for all permissible values
    A. $(\sin x - \cos x)^2 = 1 - \sin(2x)$
    B. $\cos^2(2x) - 1 = 4\sin^4 x - 4\sin^2 x$
    C. $\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1$

Solutions:

11. We can build triangles to see that $\cos A = \sqrt{40}/7$ and $\cos B = -\sqrt{33}/7$. From this is it just a matter of using the trig identities correctly; $\sin(A+B) = \frac{-3\sqrt{33}+4\sqrt{40}}{49}$ and $\cos(A+B) = \frac{-\sqrt{1320}-12}{49}$
12. We write $4\sin x - 3\cos x = r\sin(x+A)$
$$4 = r\cos A\,, \qquad -3 = r\sin A$$
$$\Rightarrow \tan A = -\frac{3}{4} \Longrightarrow A = \arctan(-0.75) = -0.64$$
$$\Rightarrow r = \frac{4}{\cos(-0.64)} = 5$$
So $5\sin(x - 0.64)$
13. Using the example above in the review $\cos(\pi/12) = \frac{1+\sqrt{3}}{2\sqrt{2}}$. Tangent is sine over cosine, so we get
$$\tan(\pi/12) = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$
14.
   A. Expand to get $\sin^2 x - 2\sin x\cos x + \cos^2 x = 1 - 2\sin x\cos x$. Finish with a double angle identity
   B. Replace $\cos(2x) = 1 - 2\sin^2 x$ since the right-hand side has no cosine. Then squaring it and subtracting 1 gives us the desired result
   C. Repeated use of double angle identity. We see that $\cos(2\cdot 2x) = 2\cos^2(2x) - 1$ since the right hand side only uses cosine and then replace $\cos(2x) = 2\cos^2 x - 1$ and expand