Chapter 6 Review

PROVING STATEMENTS

Much of this chapter is focused on using identities and good algebra to show statements are true. **If you don't know something is true you should think "HOW CAN I SHOW THIS TO BE TRUE OR FALSE?"** The only way to show the statement is unequivocally true is to use algebra and identities. To show a statement is false you need only find one instance where the statement breaks down. This goes beyond this chapter and into everyday use. Be smart and think critically how you can say something is true or false.

Our use will come from expressions, but I believe it is incredibly important you take these ideas to heart in other situations.

Big Ideas you need to know and understand:
Try things out and see if they work or do not. If yes you may be on to something and can prove it, if it doesn't work out then you know that thing you are trying needs changes or is no good.

Review questions: The following statements are true or false, if true prove it, if false show a counter-example.

1. $(a+b)^2 = a^2 + b^2$
2. $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
3. $\sin(kx) = k\sin(x)$
4. $\frac{1}{b+c} = \frac{1}{b} + \frac{1}{c}$

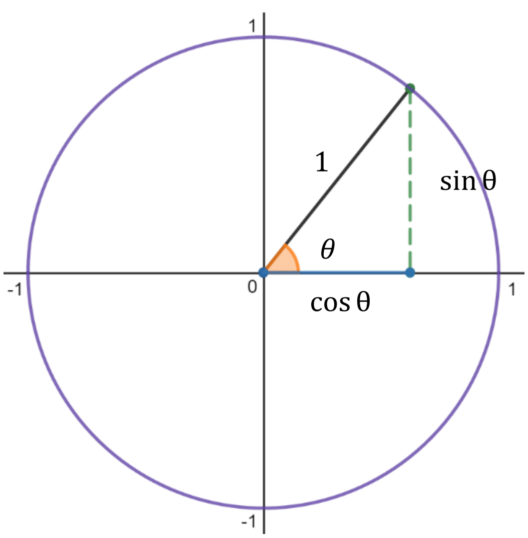
Solutions:

1. False, so long as a or b is not 0 the expression is missing the terms $2ab$.
2. True, right hand side trivially adds to the left.
3. False, only true if $k = \pm 1, 0$ else the amplitudes are not the same.
4. False, if b was very small and c was very large, then the left side would be small but the right side would be large.

THE PYTHAGOREAN IDENTITY

We use the unit circle and a right-angle triangle to prove our first identity. On the unit circle we can form a right-angle triangle with hypotenuse length 1 and height $\sin\theta$ and base $\cos\theta$. Pythagoras' Theorem states that the sum of the squares of the two smaller sides will be the square of the hypotenuse. Therefore,

$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$
$$\Longrightarrow \sin^2\theta + \cos^2\theta = 1 \ \blacksquare$$



In addition to this identity, we can construct 2 other ones by dividing by sine or cosine

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Longrightarrow 1 + \cot^2\theta = \csc^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Longrightarrow \tan^2\theta + 1 = \sec^2\theta$$

Big Ideas you need to know and understand:
Understand how to build the Pythagorean identities from a right-angle triangle just as outlined above and in class.
Recognize that when you see a trig term to a power greater than 1 and/or the number 1 in a statement that this identity will be helpful.

Review questions: Prove the following statements are true for all permissible values.

5. $\frac{2\sin^2 x+2\cos^2 x}{\csc x} = 2\sin x$
6. $\sqrt{\frac{1-\cos^2 x}{\cos^2 x}} = |\tan x|$
7. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

Solutions:

5. We factor the numerator to $2(\sin^2 x + \cos^2 x) = 2(1) = 2$. Then $2/\csc x = 2 \div (1/\sin x) = 2\sin x$ ■
6. The numerator is equivalent to $\sin^2 x$, so inside the radical is $\sin^2 x/\cos^2 x = \tan^2 x$. We know that $\sqrt{x^2} = |x|$ which completed the proof ■
7. Difference of squares so $\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = (1)(\sin^2 x - \cos^2 x)$. We can then replace $\cos^2 x = 1 - \sin^2 x$ and we have finished ■

Adding and Subtracting Angle Identities

We built the adding and subtracting angle identities using geometry. If you want to see the proof of that you can look back at 6.2 Part 1 notes, but it is not expected of you to derive the identities.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

When using the identities be careful that cosine's identity does not have the same sign on both sides of the identity (a sum of angles is a difference of terms and vice versa).

One important thing we can do with these identities is find the exact value for new angles that are of the form

$$\frac{n\pi}{12}, n \in \mathbb{Z}$$

Example: $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Additionally, from these two identities we can build the special case where the angles are the same, $\alpha = \beta$. In that case we get that $\sin(2\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$, hence

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

And we also get that $\cos(2\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$ so that

$$\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

This last identity has multiple forms because of the Pythagorean trig identity which allows us to express $\sin^2 x$ in terms of $\cos^2 x$. **IT IS IMPORTANT TO USE THE IDENTITY THAT ALLOWS YOU TO SIMPLIFY IN THE MOST EFFICIENT MANNER!**

If the expression uses sine and cosine or the fact that $\cos(2\alpha)$ is a difference of squares, then changing to $\cos^2\alpha - \sin^2\alpha$ makes sense. Otherwise we would use the others if we want all in terms of sine or cosine or we have a 1 in the expression we can get rid of.

Big Ideas you need to know and understand:
Sine and cosine addition/subtraction identities are useful when finding the sine or cosine of a new angle $(A+B)$ is hard but finding the sine and cosine of the parts (A and B) is easy.
You need to know that the double angle identities are just special cases of the general additive and subtractive identities when the angles are the same.
Use the forms of $\cos(2x)$ to suit your needs. Look around to determine what can simplify first.

Review questions:

8. If angles A and B are in quadrants I and II respectively, with $\sin A = \frac{3}{7}$ and $\sin B = \frac{4}{7}$ then determine $\sin(A+B)$ and $\cos(A+B)$.
9. Write as a single sinusoidal expression:
$$\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x$$
10. Find the exact value of $\tan(\pi/12)$
11. Prove the following is true for all permissible values
    A. $(\sin x - \cos x)^2 = 1 - \sin(2x)$
    B. $\cos^2(2x) - 1 = 4\sin^4 x - 4\sin^2 x$
    C. $\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1$

Solutions:

8. We can build triangles to see that $\cos A = \sqrt{40}/7$ and $\cos B = -\sqrt{33}/7$. From this is it just a matter of using the trig identities correctly; $\sin(A+B) = \frac{-3\sqrt{33}+4\sqrt{40}}{49}$ and $\cos(A+B) = \frac{-\sqrt{1320}-12}{49}$
9. Notice that $1/\sqrt{2} = \sin(\pi/4) = \cos(\pi/4)$. We can then replace it with
$$\cos(\pi/4)\sin x - \sin(\pi/4)\cos x = \sin(x - \pi/4)$$
10. Using the example above in the review $\cos(\pi/12) = \frac{1+\sqrt{3}}{2\sqrt{2}}$. Tangent is sine over cosine, so we get
$$\tan(\pi/12) = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$
11.
    A. Expand to get $\sin^2 x - 2\sin x\cos x + \cos^2 x = 1 - 2\sin x\cos x$. Finish with a double angle identity ■
    B. Replace $\cos(2x) = 1 - 2\sin^2 x$ since the right-hand side has no cosine. Then squaring it and subtracting 1 gives us the desired result ■
    C. Repeated use of double angle identity. We see that $\cos(2\cdot 2x) = 2\cos^2(2x) - 1$ since the right hand side only uses cosine and then replace $\cos(2x) = 2\cos^2 x - 1$ and expand ■

Trig Identities to Solve Equations

In Chapter 4 we looked at solving trig equations by following the rules: 1. Use algebra to solve for the trig term. 2. Use inverse trig and special triangles to solve for the angle. 3. Include all solutions for multiple quadrants and add $2\pi n$. 4. Continue to use algebra to solve the angle if necessary.

We are expected to do the same thing here, but we include a step at the beginning of using an identity to make all trig terms and angles the same.

In order to solve a trig equation, we need to be able to factor it into linear terms that are equal to zero. You don't always need to factor to solve, but there will be a solution that involves factoring.

Example: Find the general solution to $\sin(2x) + \sin(x) = 0$.
We need a common angle with all our terms to factor so use double angle identity

$$\Rightarrow 2\sin x \cos x + \sin x = 0 = \sin x\,(2\cos x + 1)$$
$$\Rightarrow \sin x = 0 \ \text{ OR } \ \cos x = -1/2$$

These are special triangle sides and unit circle coordinates. We have $x = \pi n$ OR $\pm 2\pi/3 + 2\pi n, n \in \mathbb{Z}$

Example: Find the general solution to $\cos^2 x + 3\sin x = 3$
We need a common trig ratio to factor so use Pythagorean identity.

$$\Rightarrow 1 - \sin^2 x + 3\sin x - 3 = 0 = \sin^2 x - 3\sin x + 2$$
$$\Rightarrow (\sin x - 2)(\sin x - 1) = 0$$
$$\Rightarrow \sin x = 1$$

We can use the unit circle to see that $x = \pi/2 + 2\pi n, n \in \mathbb{Z}$

Big Ideas you need to know and understand:
Look back at Chapter 4 to review just solving trig equations.
Your goal should be to factor the equation and to do this you need common trig ratios and common angles.

Review questions: Find the solutions for $\varphi \in [0,2\pi)$ and the general solution to the following.

12. $\cos 2\varphi + 1 = \cos\varphi$
13. $2\sin\varphi = 7 - 3\csc\varphi$
14. $\cos^2\varphi = \cot(2\varphi)\sin(2\varphi)$
15. $\sin(\pi\varphi) = \cot\left(\frac{\pi}{2}\varphi\right)$

Solutions:

12. Equivalent to $2\cos^2\varphi - \cos\varphi = 0$ so $\varphi = \pi/2 + \pi n$ OR $\varphi = \pm\pi/3 + 2\pi n, n \in \mathbb{Z}$. In the domain $\varphi = \pi/3, \pi/2, 3\pi/2, 5\pi/3$
13. Multiply both sides by $\sin x$. Then $2\sin^2\varphi - 7\sin\varphi + 3 = 0 = (2\sin\varphi - 1)(\sin\varphi - 3)$
    We get then that $\varphi = \pi/6 + 2\pi n$ OR $\varphi = 5\pi/6 + 2\pi n, n \in \mathbb{Z}$ and in the domain $\varphi = \pi/6, 5\pi/6$
14. We have $\cos^2\varphi = \frac{\cos(2\varphi)}{\sin(2\varphi)}\sin(2\varphi) = \cos(2\varphi)$ and $\sin(2\varphi) \neq 0$ We need to change to a common angle so use double angle identity. We have $\cos^2\varphi = 2\cos^2\varphi - 1$, hence $\cos\varphi = \pm 1$. However, these are non-permissible values so **NO SOLUTION**.
15. Let $A = \frac{\pi}{2}\varphi$ then we have $\sin(2A) = \cot(A)$ and we can reduce this statement to
    $$2\sin A\cos A = \frac{\cos A}{\sin A} \Rightarrow 2\sin^2 A\cos A = \cos A\,, \sin A \neq 0$$
    We get that $\cos A = 0$ OR $\sin A = \pm\frac{1}{\sqrt{2}}$. Hence, $A = \frac{\pi}{2} + \pi n$ OR $A = \frac{\pi}{4} + \frac{\pi}{2}n$ where $n \in \mathbb{Z}$.
    Substitute back in for φ and we get $\varphi = 1 + 2n$ OR $\varphi = 0.5 + n$.
    In the domain $\varphi = 0.5, 1, 1.5, 2.5, 3, 3.5, 4.5, 5, 5.5$