1. In the nuclear fission of a radioactive atom, it will decay into fission fragments and N_{avg} neutrons (the average amount per generation). These neutrons will then cause neighbouring atoms to undergo fission in a new generation and the process repeats.
   a. Determine a function for the number of neutrons being released at generation g. State the map of the function.
   b. If Uranium-235 decay into 2.4 neutrons each generation (40% of the time it releases 3, 60% of the time it releases 2), determine the number of atoms undergoing fission after 67 generations. This is slightly above the critical mass of uranium needed to create a bomb.
   c. Determine the **TOTAL** number of atoms that have undergone fission after 20 generations.
   d. Determine how many generations it takes for the **TOTAL** number of atoms to be at least 1 mole (6.023×10^{23})

2. As the altitude increases the atmospheric pressure decreases exponentially. At height h_0 (in km) the pressure is P_0 (in kPa – kilopascal) and at height h_1 the pressure is P_1.
   a. Determine a function for the pressure at height h. State the map of the function.
   b. At sea level the pressure is 101 kPa and at 4 km the pressure is 57 kPa. Determine when the pressure will be less than 5 kPa.
   c. The boundary of space is about 100 km from the surface of Earth. What is the pressure at this point in Pa (note I want the pressure in pascal NOT **kilo**pascal)?

3. The size of transistors (electronic switches) has been decreasing each year. Every n years the size decreases by p %.
   a. If in year t_0 the size of a transistor is T_0 (in μm), determine a function for the size of a transistor in year t. State the map of your function.
   b. Moore's Law states the size of commercial transistors on a circuit decrease by 25% every 2 years. In 1971 the size of commercial transistors was 10µm. Determine the year the transistor size become less than 1µm?
   c. How small does Moore's Law predict commercial transistors will become this year?

4. In a developed nation, old people die more often than young people. The probability of death is p_0 if you are a_0 years old and p_1 if you are a_1 years old. This would follow an exponential model between the ages of 2 and 100.
   a. What is the asymptote in this situation?
   b. Determine a function that determine the probability of death given a certain age. State the map of your function.
   c. In Canada, the probability of mortality in a year is 1% for people aged 60 and 30% for people aged 97. When does the probability of mortality become greater than 10%?
   d. Why would the model not be valid for ages then 2 or greater than 100?

5. Things warm exponential to the heated temperature of their surroundings, T_{amb}. We can measure something heating to be T_0 degrees at time t_0 and T_1 degrees at time t_1.
   a. Determine a function for the temperature of the object at time t. State the map of the function.
   b. We have that butter at 4°C is left out in a room with temperature is 24°C. Then, 5 hours later temperature has risen to 10°C. Determine when the temperature goes above 20°C.

c. Determine when the butter becomes room temperature.

6. In the laboratory, a population of bacteria will double every n minutes until it is restricted by the space available in the petri dish.
   a. Determine an equation that models the bacteria's population growth at some time in hours early in the growth phase when the total volume of bacteria is $v_0\ \mathrm{cm}^3$ at time t_0 (in hours). State the map of your function.
   b. If the bacteria is E.coli, it doubles every 20 minutes and the volume at 1 pm was 0.01 cm^3. Determine the time with the bacteria reach a volume of $2.5\ \mathrm{cm}^3$.
   c. Determine an equation that models the bacteria's population after this point, knowing the volume of the petri dish is $7\ \mathrm{cm}^3$ and that the bacteria were at a volume of $2.5\ \mathrm{cm}^3$ at the above time. Now we will have that the population is growing so that the available space is $\frac{2}{3}$ the size it was every 20 minutes ago.
   d. Determine when the bacteria will have a volume of $6\ \mathrm{cm}^3$.
   e. Determine when the petri dish will be full.

Solutions:

1.
   a. $N(n_{avg}, g) = (n_{avg})^{g-1} = \exp\left((g-1)\ln(n_{avg})\right)$ where N is the number of uranium atoms undergoing fission in generation g. Note that $N(1) = 1$, that is the first atom undergoing fission is generation 1. For a map we have $N: \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ as each generation must be a natural number and the **average** number of neutrons released would be a decimal since it doesn't decay into the same number each time as indicated in the next part. BUT we will only count the number of atoms released to the nearest whole number.
   b. $N(67) = 1.2 \times 10^{25}$ atoms.
   c. $\sum_{g=1}^{20}(2.4)^{g-1} = \frac{(1-2.4^{20})}{1-2.4} = 28\ 714\ 204$ atoms total
   d. $6.023 \times 10^{23} = \frac{(1-2.4^n)}{1-2.4} \Rightarrow 2.4^n = (1.4 \times 6.023 \times 10^{23} - 1) = 1.4 \times 6.023 \times 10^{23}$
   $n = \frac{23+\log(1.4\times 6.023)}{\log 2.4} = 62.9$
   So $n = 63$ generations
2.
   a. $P(h, h_0, h_1, P_0, P_1) = P_0\left(\frac{P_1}{P_0}\right)^{\frac{h-h_0}{h_1-h_0}} = P_0 \exp\left(\left(\frac{h-h_0}{h_1-h_0}\right)(\ln P_1 - \ln P_0)\right)$ where P is pressure in kPa at height h in km. For our map we have the $P: \mathbb{Q}^5 \to \mathbb{Q} \cap [0, P_{sea}]$. I would say each height would only be measured to the nearest tenth of a kilometer and pressures could definitely be decimals of a kPa (as they could be in Pa). BUT we must have that $h \geq 0$ as our model would need to be modified if we want to know the pressure as we go deeper into the Earth.
   b. $\frac{4}{\ln(57)-\ln(101)}\ln\left(\frac{5}{101}\right) = 21.0$ km so anything at or above 21.1 km will have a pressure less than 5 kPa.
   c. $P(100,0,4,101,57) = 0.000\ 06$ kPa $= 0.06$ Pa
3.
   a. $S_T(t, t_0, n, T_0, p) = T_0(1-p)^{\frac{t-t_0}{n}} = T_0 \exp\left(\left(\frac{t-t_0}{n}\right)\ln(1-p)\right)$ where S_T is the size of the transistor in micrometers and t is the year. The map of the function would be $S_T: \mathbb{N}^3 \times \mathbb{Q}^2 \to \mathbb{Q} \cap (0, T_0]$. We would have that the times should be natural numbers as they are years and $t \geq t_0$ (as we would not expect the model to predict transistor sizes in the past as they did not exist that long ago). I would think that T_0 and S_T would be rational numbers as they could get smaller than 1μm. We also have $p \in (0,1)$.
   b. $\frac{2}{\ln 0.75} \cdot \ln 0.1 + 1971 = 1987$
   c. $S_T(2021, 1971, 2, 10, 0.25) = 0.0075\ \mu\text{m} = 7.5$ nm . This is only a bit bigger than chips currently being used as the smallest are manufactured at 5 nm size.
4.
   a. The asymptote is either 0% when the age $\to -\infty$ or 100% when the age $\to \infty$. We want the chance of death to increase slowly at first and then rapidly as we age so the asymptote is 0%.
   b. $P_d(a, a_0, a_1, p_0, p_1) = p_0\left(\frac{p_1}{p_0}\right)^{\frac{a-a_0}{a_1-a_0}} = p_0 \exp\left(\left(\frac{a-a_0}{a_1-a_0}\right)(\ln p_1 - \ln p_0)\right)$ where P_d is the probability of death at age a. The map of the function would be $P_d: \mathbb{N}^3 \times \mathbb{Q}^2 \to \mathbb{Q} \cap [0,1]$. I would think that

the ages should be natural numbers and it is said in the set up that the $a \in [2{,}100]$. Also, we have that the probabilities must be in the interval $[0, 1]$.

c. $\frac{37}{\ln 0.30 - \ln 0.01} \cdot \ln 10 + 60 = 85$ so at 85 the probability of death is 10% and it increases as you age.

d. Infants are more susceptible to life threatening illnesses and sudden infant death syndrome. For the very large ages the model grows too quickly for real life and cannot go above 100%. It's likely the probability of mortality grows slower or plateaus after 100 years because these people are in care homes and have easy access to medical attention. Note that $P(100, 60, 97, 0.01, 0.3) = 40\%$.

5.

a. $T(t, t_0, t_1, T_0, T_1, T_{amb}) = (T_0 - T_{amb})\left(\frac{T_1 - T_{amb}}{T_0 - T_{amb}}\right)^{\frac{t-t_0}{t_1-t_0}} + T_{amb} = (T_0 - T_{amb}) \exp\left(\left(\frac{t-t_0}{t_1-t_0}\right)\ln\left(\frac{T_1 - T_{amb}}{T_0 - T_{amb}}\right)\right) + T_{amb}$ where T is the temperature in degrees Celsius and t is the hours after it starts to warm up. For our map we have $T: \mathbb{Q}^3 \times \mathbb{Z}^3 \to \mathbb{Z} \cap [T_0, T_{amb}]$ where I say the times can be rational (parts of an hour) and we would want that $t \geq t_0$ as this is our first measurement and the model will not work backwards. I think the temperature would be to the nearest whole degree and it could be negative so the integers makes sense.

b. $\ln\left(\frac{4}{20}\right) \cdot \frac{5}{\ln\left(\frac{14}{20}\right)} = 22.6 \Rightarrow$ just over 22 hours and 30 minutes after being taken out of the fridge.

c. Since my range is $\mathbb{Z}$, I say that $23.5 \equiv 24$, then we get that $\ln\left(\frac{24-23.5}{20}\right) \cdot \frac{5}{\ln 0.7} = 51.7$ so about over 2 days and 4 hours.

6.

a. $B_0(t, t_0, v_0, n) = v_0(2)^{\frac{t-t_0}{n/60}} = v_0 \exp\left(60\left(\frac{t-t_0}{n}\right)\ln 2\right)$ where B_0 is the volume of bacteria (in cm^3) present at time t (in hours). Note that since t is in hours and n is in minutes we need to convert them into the same unit. We would have that $B_0: \mathbb{Q}^3 \times \mathbb{N} \to \mathbb{Q} \cap [0, V_{max}]$. I would think that the times and volume could be measured in parts of an hour and parts of a cubic centimeter, but I think that the value of n would just be to the nearest minute. This model works for $t \in [t_0, t_{max}]$ where t_{max} is a time when space becomes an issue.

b. $\ln 250 \cdot \frac{20}{60 \cdot \ln 2} + 1 = 3.66$ hours, so sometime around 3: 40 pm.

c. $B_1(t) = -4.5\left(\frac{2}{3}\right)^{\frac{60}{20}(t-3.66)} + 7 = -4.5\exp\left(3(t - 3.66)\ln\frac{2}{3}\right) + 7$ where B_1 is the volume of bacteria (in cm^3) present at time t in hours ($t = 0$ is noon). We would have that $t \geq 3.66$ since we would use B_0 before this time.

d. $\ln\left(\frac{1}{4.5}\right) \cdot \frac{1}{3 \cdot \ln\frac{2}{3}} + 3.66 = 4.89$ hours, sometime around 4: 54 pm.

e. I say it's full when it's withing 1 bacteria of space. So full for me is $7\ \text{cm}^3 \equiv 6.999\ \text{cm}^3$ $\ln\left(\frac{0.001}{4.5}\right) \cdot \frac{1}{3 \cdot \ln\frac{2}{3}} + 3.66 = 10.57$ hours so sometime between 10: 30 to 11: 00 pm it should be full.