The Rational Function

The rational function below is made up of a division of polynomial functions

$$f(x) = \frac{p(x)}{q(x)} = \frac{ax^n + \cdots}{bx^m + \cdots}$$

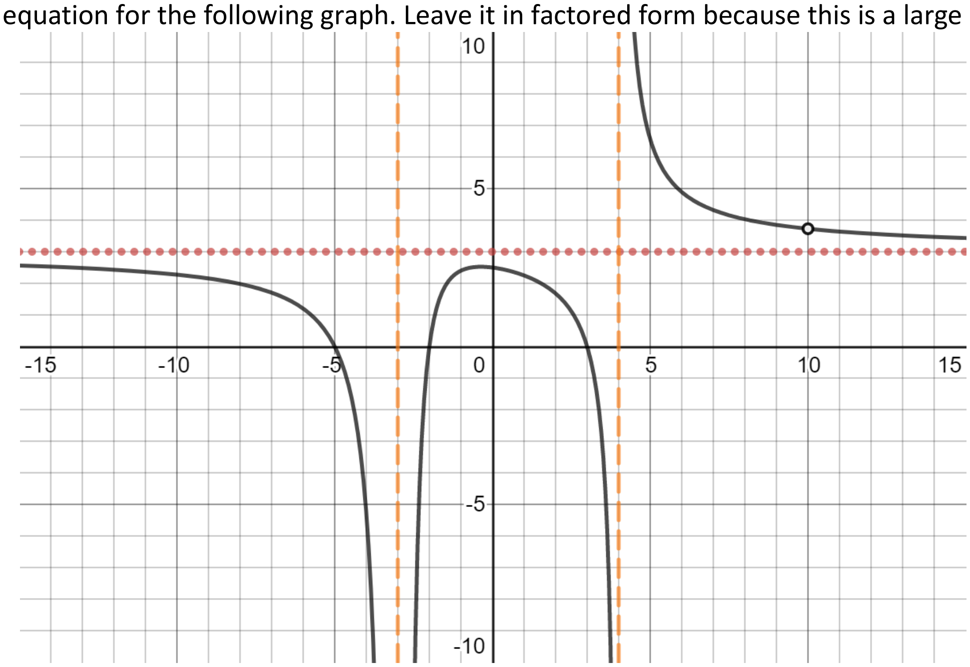
From this fraction we can deduce three important characteristics of the graph:

1. **Discontinuities** will occur when $q(x) = 0$ and it will be an **essential discontinuity** (asymptote), if $p(x) \neq 0$ OR a **removable discontinuity** (hole) if $p(x) = 0$.
2. **Zeros** will occur when $p(x) = 0$.
3. **Horizontal Behaviour** will occur at large values of x so that the function behaves just as the leading terms

$$y \approx \frac{ax^n}{bx^m} \to \text{HA} \quad \text{as} \quad x \to \pm\infty$$

Remember that the function will change sign when the x value changes over a zero or asymptote. This means that for factors with a multiplicity of 1 there will be a sign change, but not for factors with a multiplicity of 2.

Example: Make an equation for the following graph. Leave it in factored form because this is a large example.



Solution:

Identify asymptotes and the multiplicities. The multiplicity will be 2 if the function does not change sign, which occurs when $x = -3$. The other asymptote is $x = 4$. This gives us the start of the function:

$$y = \frac{1}{(x+3)^2(x-4)}$$

We can then add any holes which is at $x = 10$.

$$y = \frac{(x-10)}{(x+3)^2(x-4)(x-10)}$$

We can include zeros at $x = -5, -2$, and 3 (all have a multiplicity of 1):

$$y = \frac{(x+5)(x+2)(x-3)(x+10)}{(x+3)^2(x-4)(x+10)}$$

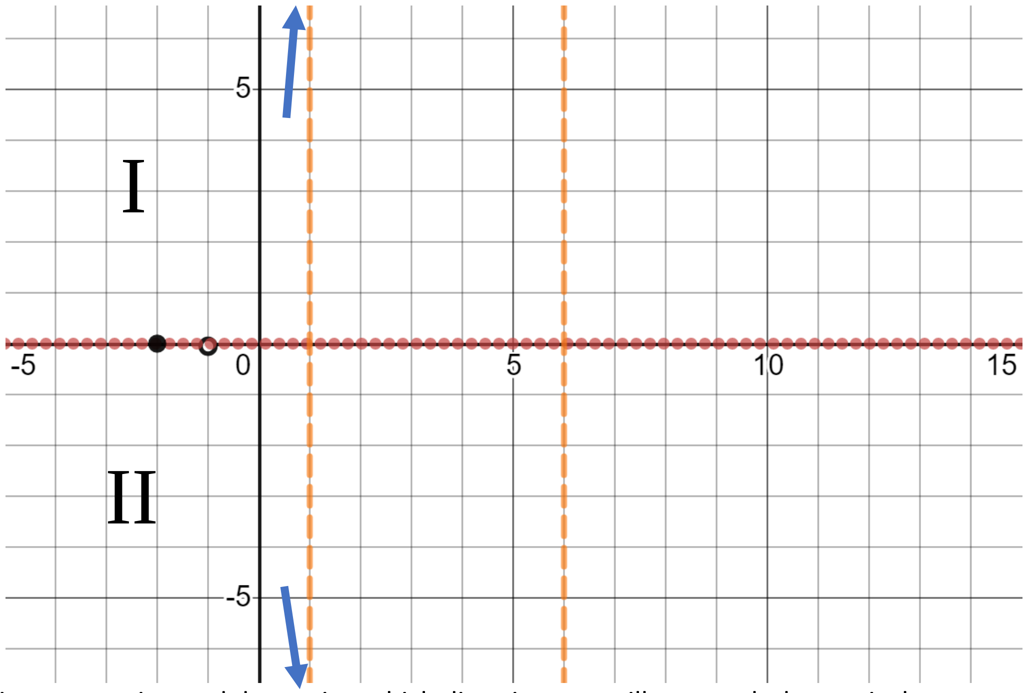
Finally, we want the function to have a horizontal asymptote at $y = 3$. Currently if we just consider the leading terms, we get $y \approx \frac{x^4}{x^4} = 1$, so adding a 3 to the numerator will give us the correct asymptote.

$$y = \frac{3(x+5)(x+2)(x-3)(x-10)}{(x+3)^2(x-4)(x-10)}$$

Example: Graph the following function

$$y = \frac{5(x+2)(x+1)}{(x+1)(x-1)(x-6)^2}$$

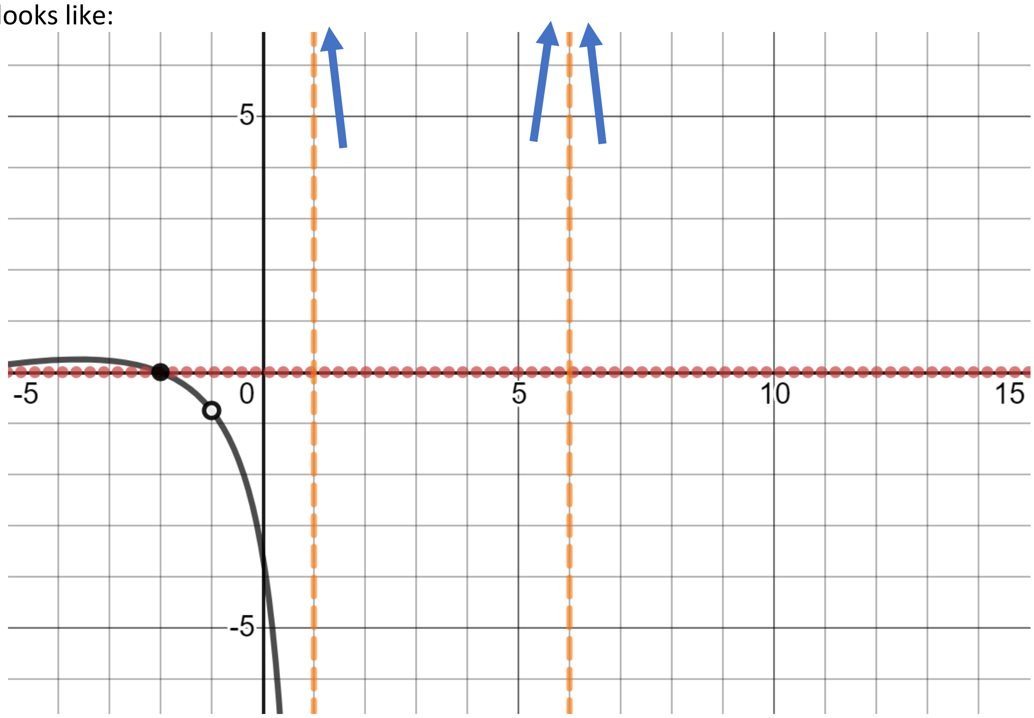
Solution:
Identify the asymptotes at $x = 1$ (multiplicity 1) and $x = 6$ (multiplicity 2), and the hole at $x = -1$.
Identify the zeros at $x = -2$ (multiplicity 1) and then the horizontal asymptote at $y \approx \frac{5x}{x^3} \to 0$. Graph the skeleton of the function.



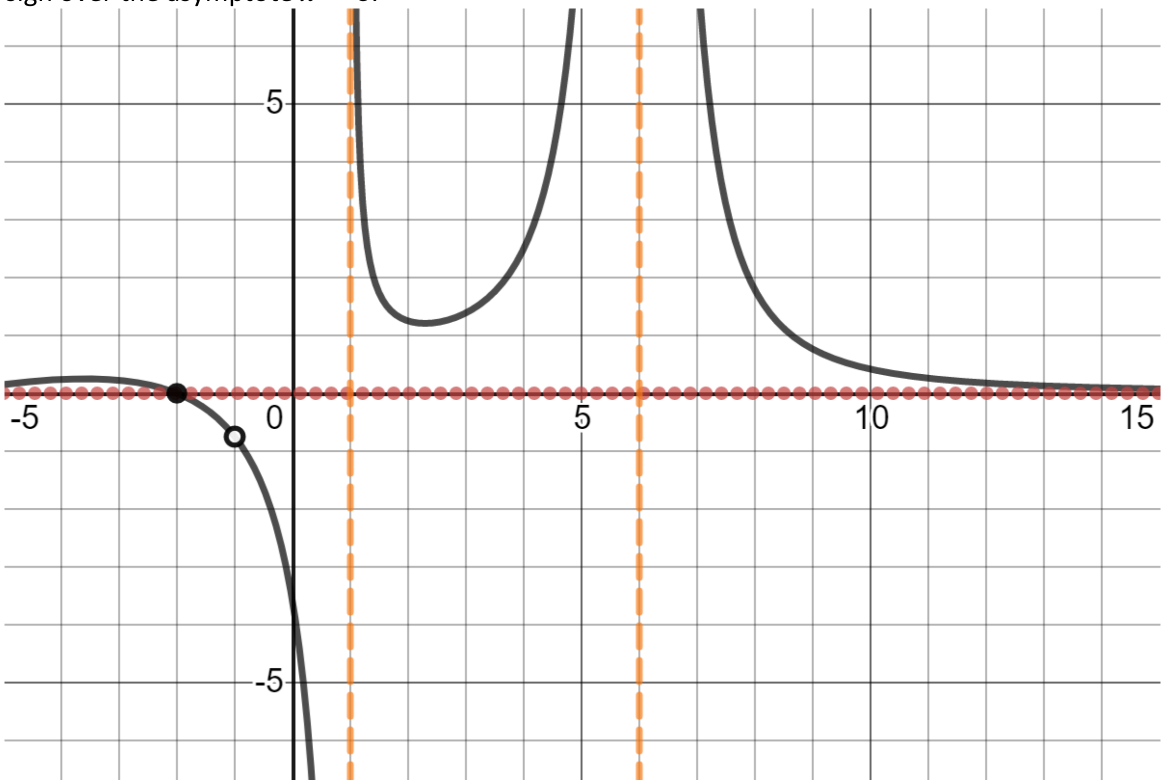
To graph it, start in some region and determine which direction you will approach the vertical asymptote (either to $+\infty$ or $-\infty$). I am looking at region I and region II as we approach the asymptote $x = 1$ from the left-hand side. Use a test value of $x = 0.9$ which is close to the asymptote but on the left. In that case we see that y will be negative because:

$$y = \frac{(+)(+)}{(-)(-)^2} = (-)$$

So, the function looks like:



We need to switch sign over the asymptote $x = 1$ and we can't cross the x-axis so the graph will turn around and then NOT switch sign over the asymptote $x = 6$.



Things we need to know and understand:

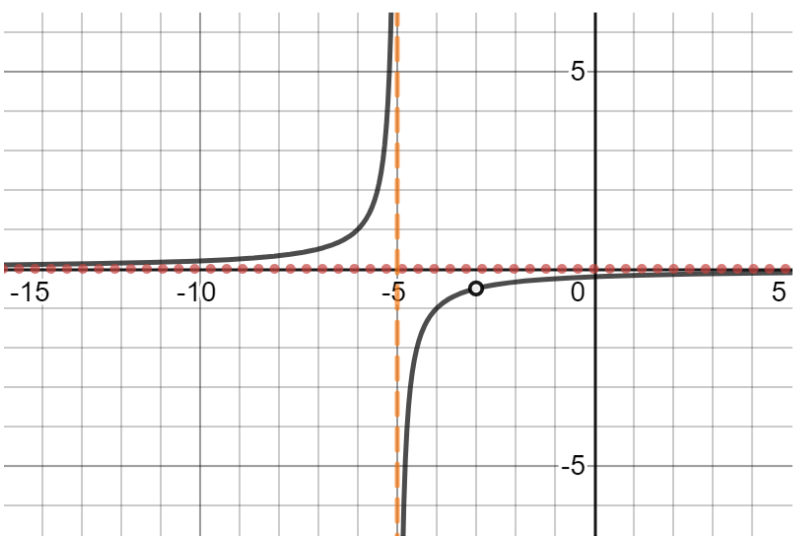
- When do discontinuities, holes, and zeros occur in a rational function.
- How to determine the horizontal behaviour by checking large values of x and only considering the leading terms.
- How to test the behaviour around an asymptote so you know what direction the graph is moving.

Review Questions:

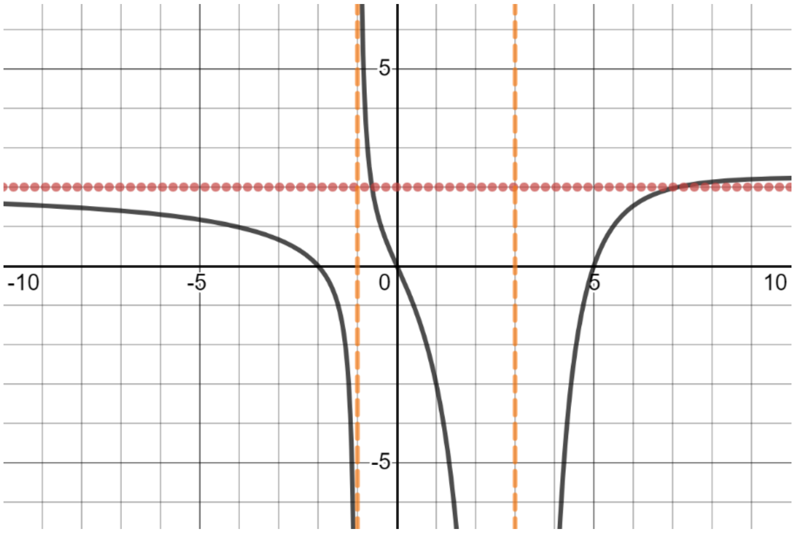
1. Determine the equation of a function with an asymptote at $x = -3$, a hole at $x = 1$, zero at $x = 5$, and a horizontal asymptote of $y = -6$.

Determine the equation of the following graphs

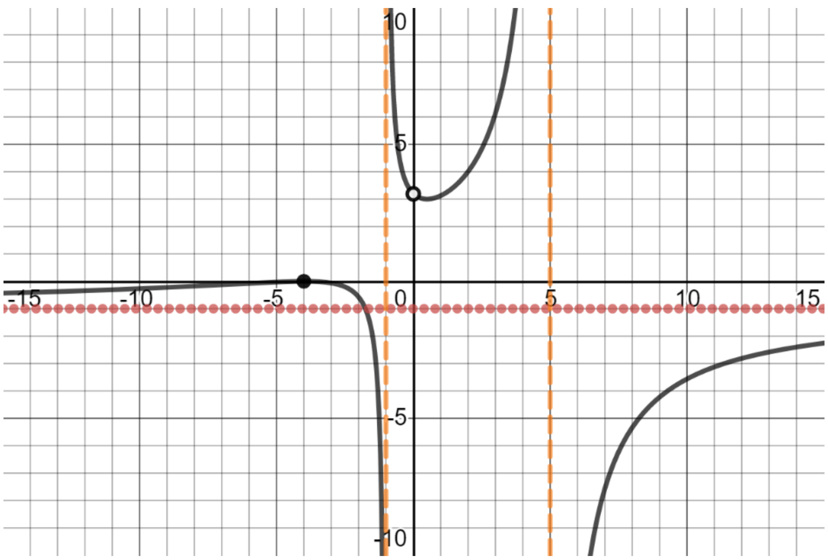
2.



3.



4.



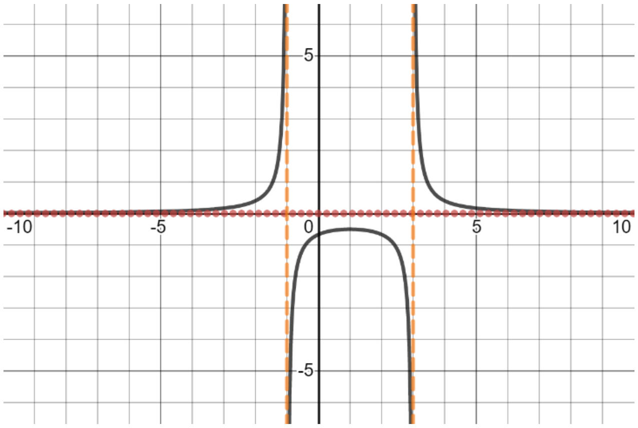
Graph the following functions

5. $y = \frac{2}{x^2-2x-3}$
6. $y = \frac{4x^2-12x+8}{x^2+x-6}$
7. $y = -\frac{2x(x-3)(x+6)}{(x+4)(x-2)^2}$
8. $y = \frac{2(x-1)^2(x+5)^2}{(x-8)(x-3)(x-1)(x+3)^2}$

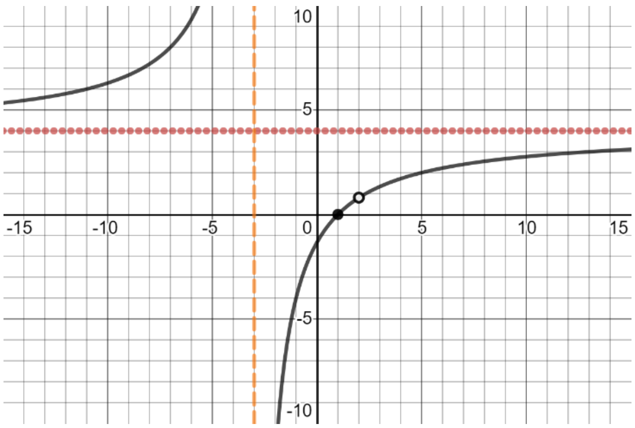
Solutions:

1. $y = -\frac{6(x-5)(x-1)}{(x+3)(x-1)}$
2. $y = -\frac{x+3}{(x+5)(x+3)}$
3. $y = \frac{2x(x+2)(x-5)}{(x+1)(x-3)^2}$
4. $y = -\frac{x(x+4)^2}{x(x+1)(x-5)}$

5.


5
-10
-5
0
5
10
-5

6.


10
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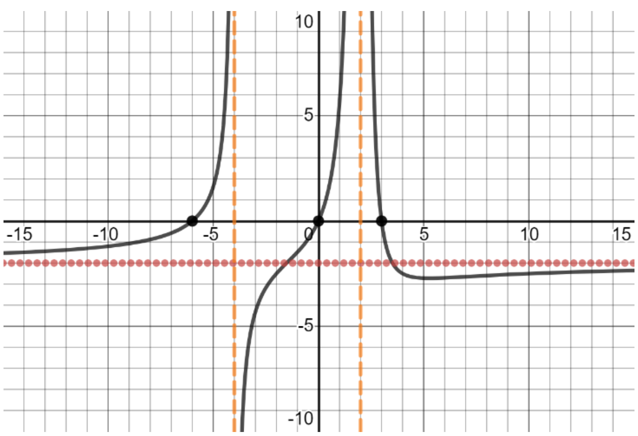
-15
-10
-5
0
5
10
15
-5
-10

7.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-15
-10
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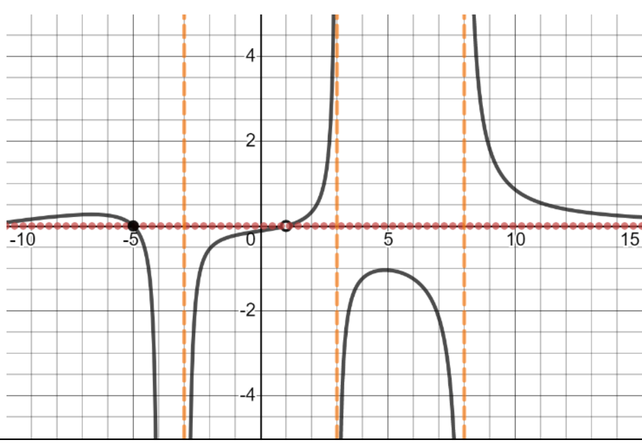
8.


4
2
-10
-5
0
5
10
15
-2
-4

Modelling Rational Applications

The major application of rational functions is that rates are additive. If you sprint 5m/s inside a train moving 20m/s, your speed relative to someone outside the train will be 25m/s or 15m/s depending on the direction you are running.

Example: We would like to model the time it takes a boat to travel down a creek a distance of d_c and then down a river a distance of d_r. In this scenario the velocity of the creek will be c and the velocity of the river will be r and the boat will be using the same amount of power in each portion so that if there was no current it would always be travelling at a velocity of v. Create an equation for the total time it takes to travel down the creek and river.

Use your equation to determine the minimum boat speed needed if the boat is to travel 2km of creek (which has a current speed of 2km/h) and 6km of river (which has a current speed of 5km/h) within 1 hour. The boat travels with the current for both legs.

Solution: The velocity of the boat in the creek is $(v + c)$ and in the river is $(v + r)$. Since velocity is distance over time, the time will be distance over velocity so

$$T = t_c + t_r$$
$$T(d_c, c, d_r, r, v) = \frac{d_c}{v+c} + \frac{d_r}{v+r}$$
$$T: \mathbb{Q}^5 \to \mathbb{Q}$$

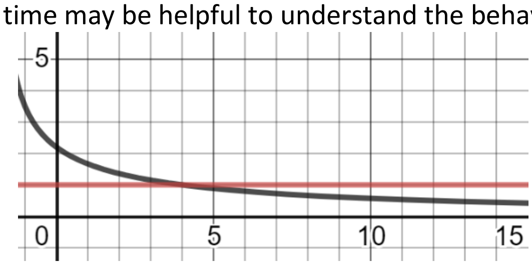
(each input must be non-negative and would likely be reported as a rational number)

Then we need to substitute values in

$$1 = \frac{2}{v+2} + \frac{6}{v+5}$$
$$v^2 + 7v + 10 = 2v + 10 + 6v + 12$$
$$v^2 - v - 12 = 0$$
$$(v-4)(v+3) = 0$$

Since we restricted our domain so that $v > 0$ we have that if the boat travels 4km/h or faster it will reach the end within 1 hour.

The graph of boat velocity versus trip time may be helpful to understand the behaviour.



Example: It takes two people 20 hours to paint a house and 1 person could do the job on their own in 32 hours. How long would it take the other person on their own?

Solution: Think about rates!

Total Speed = Speed of Person 1 + Speed of Person 2

$$\frac{1 \text{ house}}{20 \text{ hours}} = \frac{1 \text{ house}}{32 \text{ hours}} + \frac{1 \text{ house}}{t}$$

We can then solve for t

$$t = \frac{1}{\frac{1}{20} - \frac{1}{32}} = 53.3 \text{ hours}$$

Things we need to know and understand:

- Rates add together and they are expressed as something per unit of time.
- Speed is distance divided by time.

Review Questions:

9. Two people complete a job together. Their individual times to complete the job are t_1 and t_2. Determine a function for T the time together to complete the job. State it in mapping notation and describe the domain of the input variables.
10. A plane travelling at a speed p is flying against a headwind with speed w. After refueling, the plane returns with a tailwind that is 10km/h weaker than the earlier headwind. The distance of both sides of the trip is the same d. Determine a function for the time the plane is in the air (mapping notation too). Use the function to determine how fast the plane needs to fly if one way is 1600km, the headwind to start the journey is 90km/h, and the total time in the air needs to be 6 hours.
11. Three people work to complete a job and each person is half as effective as the one before them. How long will it take the three of them to complete the job relative to the most efficient person?
12. A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Determine a function for the concentration of sugar at time t in pounds per gallon. If both taps are left on forever, what will the concentration be? Is this greater or less than the initial concentration?

Solutions:

9. $\frac{1}{T} = \frac{1}{t_1} + \frac{1}{t_2} \Rightarrow T(t_1, t_2) = \frac{t_1 t_2}{t_1 + t_2}$
   $T: \mathbb{N}^2 \to \mathbb{Q}$ I think I would like the times to be natural numbers reported in minutes and then the output would be a rational number. Each time must be greater than 0. I could also see letting $T: \mathbb{Q}^2 \to \mathbb{Q}$ where the inputs are non-negative if we were thinking units would be hours.
10. $T(p, w, d) = \frac{d}{p-w} + \frac{d}{p+w-10}$ and we have $T: \mathbb{N}^3 \to \mathbb{Q}$. The plane's speed must be positive, and the wind's speed must be at least 10. The distance should also be positive. I figured natural numbers would likely be used as inputs for each. Solving we get $6 = \frac{1600}{p-90} + \frac{1600}{p+80} \Rightarrow p = 551.5$km/h
11. $\frac{1}{T} = \frac{1}{t} + \frac{1}{2t} + \frac{1}{4t} = \frac{7}{4t} \Rightarrow T = \frac{4}{7}t$ or 57% as efficient as the first person.
12. $C(t) = \frac{5+1t}{100+10t}$ and $C: \mathbb{Q} \to \mathbb{Q}$. I think we would want the input to be a rational number since we could stop filling the tank at a half minute and get a significant amount of sugar in that time. We want to see the horizontal asymptote so $C(t) \approx \frac{t}{10t} = \frac{1}{10} = 0.1$ pounds/gallon which is greater than 0.05 pounds/gallon