Chapter 2* Review

GEOMETRIC SEQUENCES AND SERIES

Geometric sequences are built when each term gets multiplied by a common ratio to build the next term

$$a_k = a_{k-1} \cdot r, \text{ and } a_1 = A$$

$$(A, Ar, Ar^2, Ar^3, \ldots, Ar^n, \ldots) = (a_1 r^{k-1})_{k \in \mathbb{N}} = (a_1 r^k)_{k=0}^{\infty}$$

We get the property of a common ratio:

$$\frac{a_{k+1}}{a_k} = r$$

If we want to add up the first n terms of a geometric sequence, then we get the following:

$$S_n = a_1 + ra_1 + r^2 a_1 + \cdots + r^{n-2} a_1 + r^{n-1} a_1$$

$$rS_n = ra_1 + r^2 a_1 + r^3 a_1 + \cdots + r^{n-1} a_1 + r^n a_1$$

And take the difference

$$(1 - r)S_n = a_1 - r^n a_1$$

$$\Rightarrow S_n = \frac{a_1(1 - r^n)}{1 - r}$$

And finally, we looked at infinite geometric series as $n \to \infty$

$$S_\infty = \frac{a_1}{1 - r}$$

Since $r^n \to 0$ when $|r| < 1$. Note that if $|r| \geq 1$ then the series will not converge and is not defined.

Big Ideas you need to know and understand:
- The summation of a series requires you to label a "first term", count how many terms are being added, and determine the common ratio.
- For infinite series, it converges when $|r| < 1$ because then the exponential $r^x \to 0$ as $x \to \infty$

Review questions:
Determine the sum of the following series
1. $S = 1 - 3 + 9 - 27 + \cdots + 6561$
2. The 4th term of the sequence is 10 and the 6th term is 22.5. What is the sum of the first 6 terms?
3. The number of new babies born into a population each year can be modelled by a geometric sequence. In the first year there are 40 babies, in the 3rd year there are 164 babies. Assuming all offspring is still alive, what is the population after 5 years?

Solutions:
1. $a_1 = 1$, $r = -3$, $n = 9 \Rightarrow S_9 = \frac{1(1 + 3^9)}{1 + 3} = 4921$
2. $a_4 = 10$, $a_6 = 22.5 \Rightarrow \frac{a_6}{a_4} = r^2 = 2.25 \Rightarrow r = \pm 1.5$. You could find a_1 since $a_4 = a_1 r^3$

   $\Rightarrow a_1 = \pm 2.\overline{962}$. However, you could just look at the series in reverse where $b_1 = 22.5$ and $r = \pm\frac{2}{3}$.

   In each case we get $S_6 = \frac{22.5\left(1 - \left(\frac{2}{3}\right)^6\right)}{1 + \frac{2}{3}} = 61.5\overline{740}$ or $12.3\overline{148} = \frac{\pm 2.962\left(1 - (1.5)^6\right)}{1 \mp 1.5}$

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3. We have $a_1 = 40$ and $a_3 = 164 = 40 \cdot r^2 \Rightarrow r = 2.02$. We want to find S_5

$$S_5 = \frac{40(1 - 2.02^5)}{1 - 2.02} = 1296 \text{ animals}$$

SIGMA NOTATION

We can define a sum using sigma notation

$$\sum_{k=n}^{N} a_k = a_n + a_{n+1} + \cdots + a_{N-1} + a_N$$

Here, k is the index that starts from a and increases by 1 until it stops at b. We add each result to each other

This allows us to rewrite S_n in terms of sigma notation

$$S_n = a_1 + a_1 r + \cdots a_1 r^{n-1}$$

$$= \sum_{k=0}^{n-1} a_1 r^k$$

$$= \frac{a_1(1 - r^n)}{1 - r}$$

Big Ideas you need to know and understand:
- Sigma notation is just a condensed way to write summations
- The index k is a dummy variable and could be labeled anything

Review questions:
Evaluate the following and change their index to start at 0.
4.
$$\sum_{k=1}^{10} \frac{(1.3)^k}{3}$$

5.
$$\sum_{k=-3}^{5} (-2)^k \cdot \frac{4}{5}$$

6.
$$\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k \cdot 8$$

For 7 and 8, write them using sigma notation with an index that starts at 0 and at 1.
7. $3 + 6 + 12 + 24 + \cdots + 768$
8. $1215 - 405 + 135 - 45 + \cdots - \frac{5}{9}$
9. Show the following is true:
$$\sum_{k=-N}^{N} k^2 = 2 \sum_{k=0}^{N} k^2$$

Solutions:

4. $a_1 = \frac{1.3}{3},\; r = 1.3,\; n = 10 \Rightarrow S_{10} = \frac{\left(\frac{1.3}{3}\right)(1-1.3^{10})}{1-1.3} = 18.468\ldots$

$$\sum_{k=0}^{9} \frac{(1.3)^{k+1}}{3}$$

5. $a_1 = -\frac{1}{10},\; r = -2,\; n = 9$ (from -3 to 5 is nine terms) $\Rightarrow S_9 = \frac{\left(-\frac{1}{10}\right)(1+2^9)}{1+2} = -17.1 =$

$$\sum_{k=0}^{8} (-2)^{k-3} \cdot \frac{4}{5}$$

6. $a_1 = \frac{9}{2},\; r = \frac{3}{4},\; n = \infty \Rightarrow S_\infty = \frac{4.5}{1-0.75} = 18$

$$\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k+2} \cdot 8$$

7. $\sum_{k=0}^{8} 3 \cdot 2^k = \sum_{k=1}^{9} 3 \cdot 2^{k-1}$

8. $\sum_{k=0}^{7} 1215 \cdot \left(-\frac{1}{3}\right)^k = \sum_{k=1}^{8} 1215 \cdot \left(-\frac{1}{3}\right)^{k-1}$

9. We have that:

$$\sum_{k=-N}^{N} k^2 = (-N)^2 + (-N+1)^2 + \cdots + (-1)^2 + 0^2 + 1^2 + \cdots + (N-1)^2 + N^2$$

$$= N^2 + (N-1)^2 + \cdots + 1^2 + 0 + 1^2 + \cdots + (N-1)^2 + N^2$$
$$= 2N^2 + 2(N-1)^2 + \cdots + 2(1)^2 + 0$$
$$= 2\sum_{k=1}^{N} k^2 + 0 = 2\sum_{k=0}^{N} k^2$$

Chapter 7 and 8 Review

GRAPHING THE EXPONENT

The basic exponential function is $f(x) = b^x$ for some base $b > 0$ and $b \neq 1$. We don't allow the base to be negative because then the function f is undefined infinitely often (taking even roots of a negative number is not allowed over the reals), and if the base is 0 or 1 then the function is a straight line (if $b = 0$ we also have the problem that 0^0 is not well defined – a problem for calculus).
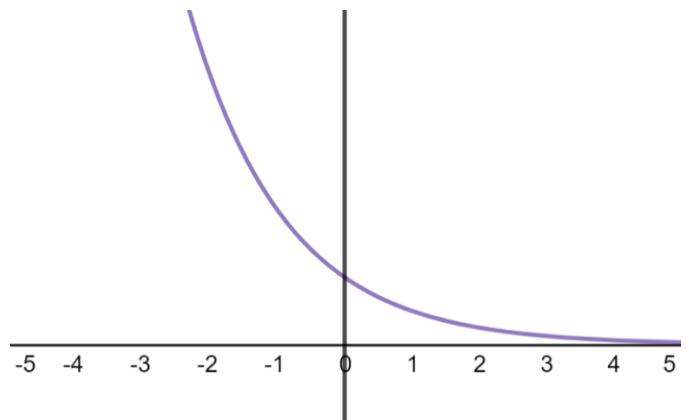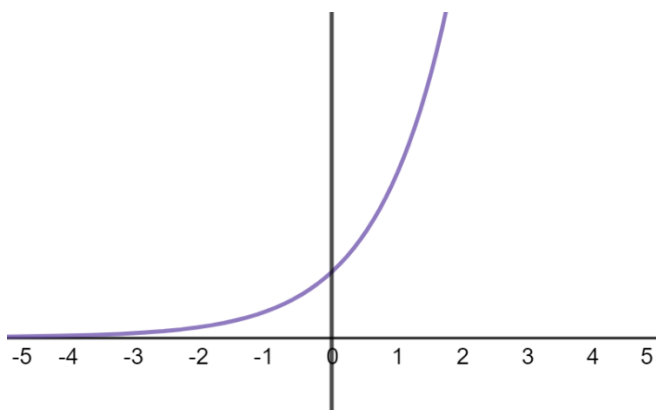
Review

The two basic shapes (first if $b > 1$, second if $b \in (0,1)$) of the exponential. They both have a horizontal asymptote at $y = 0$ and a y-intercept of 1 (because $b^0 = 1$ for any base) and they will pass through the points $(1, b)$ and $\left(-1, \frac{1}{b}\right)$.

When graphing a **transformation** of the exponential, say $g(x) = af\left(\frac{1}{T}(x - c)\right) + d$ where $f(x) = r^x$, understand that the transformations change the characteristics of the function in different ways. That is:
- The asymptote is only affected by vertical shifts (ie the asymptote is d)
- The point $(0,1)$ vertical stretches and shifts (and horizontal shifts if any). It will move to $(c, a + d)$ so you can identify the shift left right and the stretch since you already know d. Remember this point is usually a distance 1 from the asymptote and the vertically stretch changes this distance.
- The point $(1, b)$ is affected by everything but it is usually 1 unit to the left of $(0,1)$ so horizontal stretches will change this distance.

Big Ideas you need to know and understand:
You need to know the three basic characteristics of the exponential so you can effortlessly graph the basic exponential.
When graphing a transformed function, understand why the asymptote is only affected by vertical shifts and find the image of $(0,1)$ and $(1, b)$
When building the equation to a graph, start with the asymptote and then identify the vertical stretch and horizontal shift. The basic function is found from the heights of the points to the asymptote and the horizontal distance between them.

Review questions: Graph the following functions
10. $g(x) = -2^{\frac{x+2}{2}} + 3$
11. $g(x) = 3 \cdot \left(\frac{2}{3}\right)^{\frac{x-1}{4}} - 2$
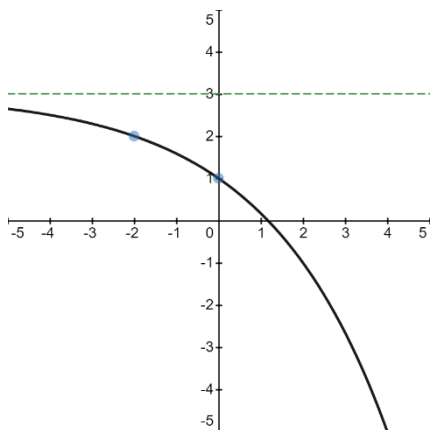
Build functions in base e that pass through the given points and have the stated asymptote.
12. $(6, 8)$ and $(8, 5)$ with a horizontal asymptote of $y = 3$
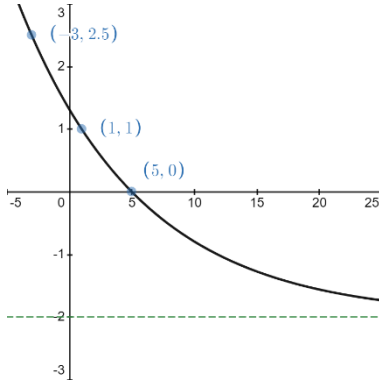13. $(-9, -9)$ and $(3, 5)$ with a horizontal asymptote of $y = 10$

Solutions:
10.

Review

11.

12. $f(x) = 5 \cdot \exp\left(\ln\frac{2}{5} \cdot \frac{x-6}{2}\right) + 3$

13. $f(x) = -19 \cdot \exp\left(\ln\frac{5}{19} \cdot \frac{x+9}{12}\right) + 10$

MODELLING FUNCTION BEHAVIOUR

To model an exponential function, you need to remember that the growth between points is because of multiplication not addition. The function will grow or decay at this value r where every T units of time we multiply the amount we have by r. We then get the total amount, A, at time t is:

$$A(t) = r^{t/T}$$

If we need to adjust the asymptote, then we think about it in terms of transformations. Let d be the maximum or minimum of the model so the equation is then:

$$A(t) = r^{t/T} + d$$

Pick one point on your graph and use that as the image of $(0,1)$. Then we can look at how far away our point is from the asymptote. Typically, it is 1 unit away so if it is a units away that is my vertical stretch:

$$A(t) = a \cdot r^{t/T} + m$$

Finally, our point is typically on the y-axis. We can see how much it was shifted left or right by using the x-coordinate.

$$A(t) = a \cdot r^{(t-c)/T} + m$$

Example: A turkey is cooking in an oven at 325°F (the asymptote). At noon the temperature is 75°F and then 2 hours later the temperature of the turkey is 112.5°F. Make an equation to determine the temperature at 4pm and change your equation to base e toto determine when the temperature is 175°F (done).

Solution. We know we pass through $(0, 75)$ and $(2, 112.5)$ using $(time, temp)$. We also know the asymptote is 325 (degrees Fahrenheit). Our equation must be

$$T(t) = a \cdot r^{t/T} + 325$$

To build the basic part, the first point is 250 units away from the asymptote and the second is 212.5 units away from the asymptote. It takes us 2 hours to change from $250 \rightarrow 212.5$. Our $r = \frac{212.5}{250} = 0.85$ and $T = 2$

$$T(t) = a \cdot (0.85)^{t/2} + 325$$

Finally, the point $(0,75)$ is still on the y-axis so no shift left or right, but it is 250 units away from the asymptote (the vertical stretch) and it's below the asymptote so there was a reflection over the x-axis $\Rightarrow a = -250$

$$T(t) = -250 \cdot (0.85)^{t/2} + 325$$

So, $T(4) = 144.4°$F. To change it to base e we need to write $0.85 = e^k$ where $\ln 0.85 = k$

$$T(t) = -250 \cdot (e^k)^{t/2} + 325$$
$$= -250 \cdot e^{\frac{kt}{2}} + 325$$

$$175 = -250 \cdot e^{\frac{kt}{2}} + 325$$
$$\frac{175 - 325}{-250} = e^{kt/2}$$
$$\ln 0.6 = \frac{kt}{2}$$

$$t = \frac{2\ln 0.6}{k} = \frac{2\ln 0.6}{\ln 0.85} = 6.3 \approx 6\!:\!20\text{pm}$$

Big Ideas you need to know and understand:
- Understand how exponential functions grow by repeated multiplication.
- How to identify the asymptote from the context of the problem thinking about what happens as $t \to \infty$ or $t \to -\infty$

Review Questions:
See other hand out.

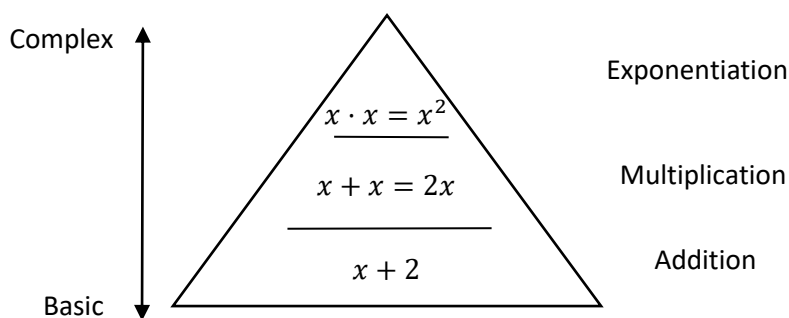**Log Laws**

We discussed the use of 4 major log laws:

Product: $\log_b(M \cdot N) = \log_b M + \log_b N$ **Power**: $\log_b x^n = n \cdot \log_b x$

Quotient: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ **Change of base**: $\log_b a = \frac{\log_X a}{\log_X b}$

The big idea is that logarithms make **INSIDE** operations less complex **OUTSIDE** as illustrated below:

Complex

Basic

$x \cdot x = x^2$

$x + x = 2x$

$x + 2$

Exponentiation

Multiplication

Addition

Review

The power law takes exponents, as in $\log x^3$, and turns it into a multiplication statement, $3 \cdot \log x$. The product/quotient law takes multiplication/division like $\log 3x$, and turns it into addition or subtraction, $\log 3 + \log x$. **The log of an operation is converted into the operation of logs that is directly below it in simplicity**.

These laws are used to simplify and expand logarithms depending on the circumstance. Most often, we will want to simplify logarithms and bring it together under a single logarithm (bring coefficients up as a power and bring sums of logs together as a product). Less often, we will break a single logarithm apart into pieces because we can evaluate the smaller parts (this is where we bring powers down and change a log of a product into a sum of logs).

The change of base is used primarily to evaluate logarithms in terms of the common log or natural log, that is

$$\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

However, we will often use change of base to get a common base among a set of logarithms so that we can implement product or quotient law.

Things we need to know and understand:
- When simplifying we need to use power law first to get rid of coefficients, then we can use product/quotient laws to bring a sum/difference of logs together as a single log.
- When expanding we need to first expand as a sum/difference using product/quotient laws and then bring down any exponents using power law.

Review Questions:
14. Why does $\log(M + N) \neq \log M + \log N$ and $\log M \cdot \log N \neq \log M + \log N$?
15. Simplify the following as a single logarithm: $\frac{2}{3}\log_5 x - 4\log_{25} y - 3\log_5 z$

16. If $\log_7 2 = a$, $\log_7 3 = b$, and $\log_7 5 = c$, then determine $\log_7\left(\frac{\sqrt{40}}{63}\right)$ in terms of a, b, c.
17. Write $g(x) = \log_2 x^3 - 3$ as a common log of x (some transformation of $f(x) = \log x$)

Solutions:
14. For the first part the operation "log" does not have the distributive property multiplication has. Basically, multiplication is the only operation that *does* have distributive property (not even division has all of it) so don't do it with logs! For the second part if $\log M = A$ and $\log N = B$ ask yourself does $A \cdot B = A + B$ for every single $A, B \in \mathbb{R}$? No, of course not. Multiplication is not addition. **DO NOT MAKE THESE COMMON MISTAKES!**
15. $\log_5\left(\frac{x^{\frac{2}{3}}}{y^2 z^3}\right)$
16. $1.5a - 2b + 0.5c - 1$
17. $g(x) = \frac{3}{\log 2} \cdot \log(x) - 3 \approx 10\log x - 3$

Solving Exponents and Logs Algebraically

When we combine logarithms together, we need to realize that in the simplification process things will cancel or multiply together which can change the sign. Since we cannot take the log of a negative, this will alter the domain. For example, in $2\log x$ we have that $x > 0$, but this is technically the same function as $\log x^2$ and here $x \neq 0$. The domains do not match so we need to state they are both $x > 0$.

When you have a set of logs, each of them must be satisfied and have a positive argument. To restrict the domain for a series, write them all out and see what is common with everything.

Review

Solving an equation that involves logarithms is just a process of simplifying it to a single logarithm, change to exponential, and then solve the resulting equation as you have done many times before. Alternatively, you can solve both sides of an equation to a single logarithm and then only consider the input of the log since it is one-to-one.

Example: Solve
$$2\log(x+1) - \log x = \log(x-2)$$

Solution: Note the domain is $x > 2$. We can first move everything to one side and then use log laws
$$2\log(x+1) - \log x - \log(x-2) = 0$$
$$\log\left(\frac{(x+1)^2}{x(x-2)}\right) = 0$$

Changing to exponential form we get
$$\frac{(x+1)^2}{x(x-2)} = 10^0 = 1$$

So that
$$(x+1)^2 = x(x-2)$$
$$x^2 + 2x + 1 = x^2 - 2x$$
$$x = -\frac{1}{4}$$

There is **NO SOLUTION** because $x > 2$.

Remember that the logarithm is the inverse of the exponential so when we need to solve for an exponent you will need to change it into logarithm form and vice versa if you are solving for the argument in a logarithm.

Things we need to know and understand:
- How to determine the domain of a sum of logs.
- How to solve for a variable in an exponent and how to solve for a variable inside a log.

Review Questions:
18. What is the domain of $f(x) = \log(x+5) + \log((x+1)(x-1)) - \log(x(x-3))$
19. What is the solution to $\log_2(x+3) = \log_4((x-2)(x+3)) + 1$
20. What is the solution to $\log((x-5)(x-2)) + \log((x+3)(x+1)) = \log((x-5)(x+3))$

Solutions:
18. $-5 < x < -1$ OR $x > 3$
19. $x = \frac{11}{3}$ (domain is $x > 2$)
20. $x = \frac{1\pm\sqrt{13}}{2} \approx -1.3$ and 2.3 (domain is $x < -3$ OR $x > 5$) so NO SOLUTION