5.3 - Properties of the definite integral and Mean Value Theorem

Mr. Guillen's AP Calculus

We definied the area under the curve $f(x)$ on the interval $[a,b]$ as:

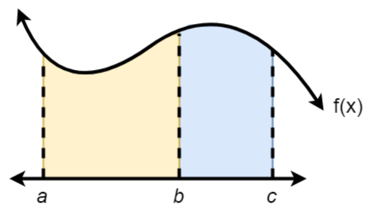
$$\text{Area} = \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x$$

Using this basic idea that the integral is the area we can define some important properties:

i. $\int_a^a f(x)dx = 0$, since the interval has a length of 0 the area is 0.

ii. $\int_a^b f(x)dx = -\int_b^a f(x)dx$, because if we take the area moving from b to a, the partions of the Riemann sum will have $\Delta x < 0$ and so $dx < 0$.

iii. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$, implying the areas add together.



iv. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

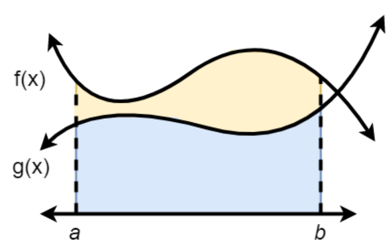
v. $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx, k \in \mathbb{R}$

Proof. I will prove (v) and leave (iv) as an exercise as it follows a very similar idea. If we use the definition of the integral as a Riemann sum the proof is fairly starightforward.

$$\begin{aligned}\int_a^b k\cdot f(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} k\cdot f(c_i)\Delta x, \qquad \Delta x = \frac{b-a}{n}\\ &= k\cdot \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x\\ &= k\cdot \int_a^b f(x)dx\end{aligned}$$

$\square$

vi. if $f(x) \geq g(x)\ \forall\ x \in [a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$



vi. $\min(f)\cdot(b-a) \leq \int_a^b f(x)dx \leq \max(f)\cdot(b-a)$, the proof is left for the students.

Example 1

Let $\int_0^2 f(x)dx = 3, \int_2^3 f(x)dx = 1$, and $\int_0^3 g(x)dx = -1$. Then we have that:

a. $\int_0^3 f(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx = 4$

b. $\int_0^3 (f(x) + g(x))dx = \int_0^3 f(x)dx + \int_0^3 g(x)dx = 3$

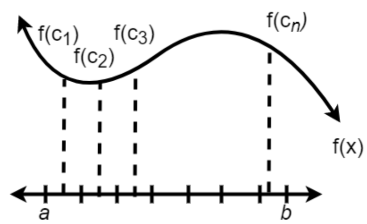
c. $\int_3^0 5g(x)dx = -\int_0^3 5g(x)dx = -5\int_0^3 g(x)dx = 5$

Mean Value Theorem

If $f(x)$ is continuous on the interval $[a, b]$ then $\exists\ c \in [a, b]$ such that $f(c) = \text{avg}(f)$, where $\text{avg}(f)$ is the average value of $f(x)$.

Proof. This should sort of make sense intuitively. The function is continuous so there are no holes or jumps on the interval we are looking at; therefore, it should go through all values from the lowest value of $f(x)$ to the largest. To formalize this we need to properly define what $\text{avg}(f)$ is.

Consider how we defined area under a curve by partitioning the interval and taking finer and finer partitions so that the limit approaches a fixed value. We will do the same with averages. Begin by making a partition by dividing the interval into n even sections such that $\Delta x = \frac{b-a}{n}$. For each subinterval take some $c_k \in (x_k, x_k + 1)$ that will act as a point for us to sample $f(x)$. To find the average we will have to add up all these sample values of $f(c_k)$ and divide by the number of samples, n.



$$
\begin{aligned}
\text{avg}(f) &\approx \frac{f(c_1) + f(c_2) + \ldots + f(c_n)}{n} \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} f(c_k) \\
&= \frac{\Delta x}{b-a} \cdot \sum_{k=1}^{n} f(c_k) \\
&= \frac{1}{b-a} \cdot \sum_{k=1}^{n} f(c_k)\Delta x \\
\text{avg}(f) &= \frac{1}{b-a} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x \\
&= \frac{1}{b-a} \cdot \int_a^b f(x)dx
\end{aligned}
$$

Let's stop there and make sense of the final statement, $\text{avg}(f) = \frac{1}{b-a} \cdot \int_a^b f(x)dx$, which can be interpreted as:

$$\text{Average Height} = \frac{\text{Area}}{\text{Width}}$$

And this should make sense and hopefully reduce a hard problem to an almost trivial statement.

Moving forward we can actually prove the Mean Value Theorem. We have by integral properties that

$$\min(f) \cdot (b-a) \leq \int_a^b f(x)dx \leq \max(f) \cdot (b-a)$$
$$\min(f) \leq \frac{1}{b-a} \cdot \int_a^b f(x)dx \leq \max(f)$$
$$\min(f) \leq \text{avg}(f) \leq \max(f)$$

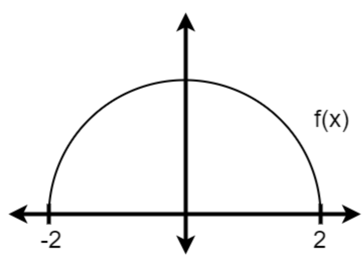
Since f is continuous, $f(x)$ must be equal to every value on $[\min(f), \max(f)]$ including the average value of f. Therefore, there must be some $c \in [a,b]$ such that $f(c) = \text{avg}(f)$. □

Notice that the Mean Value Theorem tells us nothing about finding this value of c, only that it exists. The big thing is that it requires the function to be continuous, so in general this theorem does not hold for discontinious functions.

That said, we can use the idea we developed for the average value even before we are able to compute integrals.

Example 2

What is the average value of the function $f(x) = \sqrt{4-x^2}$ on the interval $[-2, 2]$?



Notice that the function is a semi circle on that interval and so $\int_{-2}^{2} \sqrt{4-x^2}dx = \frac{1}{2} \cdot (\pi(2)^2) = 2\pi$ since we know how to compute the area of a circle. Therefore, the average value of this function is:

$$\text{avg}(f) = \frac{1}{4} \cdot \int_{-2}^{2} \sqrt{4-x^2}dx = \frac{\pi}{2} \approx 1.57$$