Derivative Rule Practice

1. Explain how we derived the four big trig rules:

a. Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

b. Sum/Difference Rule

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

c. Product Rule

$$\frac{d}{dx}(fg) = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}$$

d. Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$$

2. Find the derivative of these functions and then find the derivative of that function.

   a.

$$y = \frac{x^2 + 5x - 1}{x^2}$$

   b.

$$y = \frac{x^2 + 3}{12x} \cdot \frac{1}{x^3}$$

3. Find equations for horizontal tangents to the curve $y = x^3 - 3x - 2$

4. The curve $y = ax^2 + bx + c$ passes through the point $(1,2)$ and is tangent to the line $y = x$ at the origin. Determine a, b, c

5. The product rule says that

$$\frac{d}{dx}(fg) = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}$$

a. What is the rule for a product of three functions?

$$\frac{d}{dx}(fgh)$$

b. What is the rule for a product of four functions?

c. What is the rule for a product of n functions?

6. If gas in a closed container (or system) then the pressure, volume and temperature of the gas can be accurately modelled by the Van der Waals equation:

$$\left(P - \frac{a}{V_m^2}\right)(V_m - b) = RT$$

Where

- P is the pressure (measured in atmospheres, atm)
- V_m is the molar volume (the volume per unit of gas and is measured in litres per mole, L/mol)
- T is the temperature (measured in kelvin, K)
- R is the universal gas constant ($0.08205\,\frac{\text{atm}\cdot\text{L}}{\text{mol}\cdot\text{K}}$)

And a and b are constants related to the gas.

a. Assume that volume of the gas is constant. Determine the rate of change of temperature as pressure changes.

b. Assume that temperature is constant. What is the rate of change of pressure as volume changes?

c. Based on the equation, what does b represent?

Solutions:

1. See class notes
2. a.

$$y' = \frac{(2x+5)(x^2) - 2x(x^2+5x-1)}{x^4} = \frac{2x^3+5x^2-2x^3-10x^2+2x}{x^4} = \frac{-5x^2+2x}{x^4} = \frac{-5x+2}{x^3}$$

$$(y')' = \frac{-5x^3 - 3x^2(2-5x)}{x^6}$$

b.

$$y' = \frac{2x(12x^4) - 48x^3(x^2+3)}{144x^8} = \frac{24x^5 - 48x^5 - 144x^3}{144x^8} = -\frac{24x^5+144x^3}{144x^8} = -\frac{x^2+6}{6x^5}$$

$$(y')' = \frac{30x^4(x^2+6) - 2x(6x^5)}{36x^{10}}$$

3. $y' = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$, so a horizontal tangent at $(1,-4)$ and $(-1,0)$
   Equations $y = -4$ and $y = 0$
4. $y' = 2ax + b = 1$ at $(0,0)$, hence $b = 1$. The curve passes through the origin so $c = 0$ and since it passes through $(1,2)$ we have $2 = a + 1$ so $a = 1$. The curve is $y = x^2 + x$
5. a. For $(fgh)'$ set $fg = F$. Then use product rule to find $(Fh)' = F'h + Fh'$, but $F' = f'g + g'f$ so we get $(fgh)' = f'gh + fg'h + fgh'$
   b. The same idea will work for four functions.
   $$(fghk)' = f'ghk + fg'hk + fgh'k + fghk'$$
   c. You can extend these ideas for n functions and get that
   $$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n{}'$$

   If you want a nice closed form of this, use sigma and pi notation
   $$(f_1 f_2 \cdots f_n)' = \sum_{k=1}^{n} f_k{}' \prod_{\substack{i \in A \\ i \neq k}} f_i$$
   Where $A = \{1, 2, 3, \dots, n\}$
6. a. We want to find $\frac{dT}{dP}$. Solve for T
   $$T = \left(P - \frac{a}{V_m^2}\right) \cdot \frac{(V_m - b)}{R}$$
   $$\frac{dT}{dP} = \frac{(V_m - b)}{R}$$
   b. We want to find $\frac{dP}{dV_m}$. Solve for P
   $$P = \frac{RT}{V_m - b} + \frac{a}{V_m^2}$$
   $$\frac{dP}{dV_m} = -\frac{RT}{(V_m - b)^2} - \frac{2a}{V_m^3}$$
   c. b subtracts from the total volume so it represents the volume of the particle.