Higher Order Differentiation

Goal:
- Can use derivative rules (product, quotient, and chain) to find the second, third and n^{th} derivative of a polynomial function.
- Can use implicit differentiation to find d^2y/dx^2
- Can use proper notation to describe second, third and n^{th} derivatives

Terminology:
- Order of the derivative

Reminder:
- Quiz on Thursday November 14
- Test on Wednesday November 20

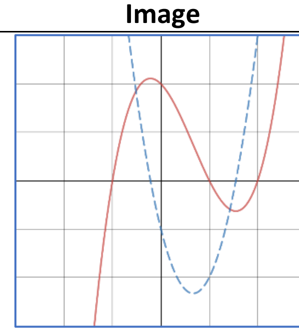
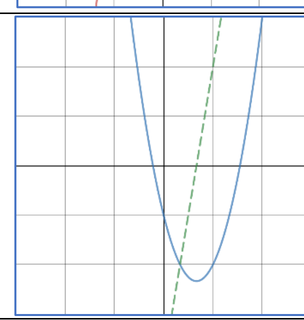
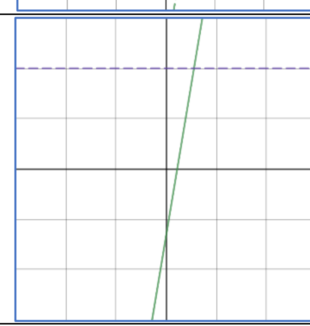
There is nothing new to discuss today. We have talked about derivative rules and how to take a derivative implicitly.

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}(f \cdot g) = g \cdot \frac{df}{dg} + f \cdot \frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2} \qquad \frac{d}{dx}f(u) = \frac{df}{du} \cdot \frac{du}{dx}$$

The big goal of the day is that you can take the derivative of a derivative, but you already did that with earlier review. All I am going to do is formally state the notation for these derivatives.

If $y = f(x)$, then we can describe the derivative of this function:

Derivative Notation	**Meaning**	**Image**
$\frac{dy}{dx}$ $\qquad y'$		
		
		

Derivative Notation	Meaning	Image

All I want you to do is practice finding derivatives using chain rule and implicit differentiation. Remember that simplifying makes your life easier and we only expand if we plan on simplifying later.

Practice

1. Determine the second derivative rules for:
   a. Power rule

$$\frac{d^2}{dx^2}x^n$$

   b. Product rule

$$(f \cdot g)''$$

c. Quotient rule

$$\left(\frac{f}{g}\right)''$$

d. Chain rule

$$\frac{d^2}{dx^2}f(u)$$

2. Find an expression for $d^n y/dx^n$ given:

$$y = x \cdot f(x)$$

3. Find an expression for $y^{(n)}$ given:

$$y = g(2x)$$

4. Find $y^{(4)}$ given

$$y = h(x^2)$$

5. Find dy/dx, d^2y/dx^2 and d^3y/dx^3 for the following relation:

$$y - y^3 = x^2$$

Practice Problems: 2.8: # 1-2 (do what you need), 3-5, 7, 9

 # 6, 8, 10

Solutions:

1.
   a. $\frac{d^2}{dx^2}x^n = n(n-1)x^{n-2}$
   b. $(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$
   c. $\left(\frac{f}{g}\right)' = \frac{(f''g - g''f)g^2 - 2g \cdot g'(f'g - g'f)}{g^4}$
   d. $\frac{d^2}{dx^2}f(u) = \frac{d}{dx}\left(\frac{df}{du} \cdot \frac{du}{dx}\right) = \frac{d^2 f}{du^2} \cdot \left(\frac{du}{dx}\right)^2 + \frac{d^2 u}{dx^2} \cdot \frac{df}{du}$

      Or using prime notation:

      $\left(f(u)\right)'' = (f'(u) \cdot u')' = f''(u) \cdot (u')^2 + u'' \cdot f'(u)$

      Hopefully you are separating the derivative $\frac{d^2}{dx^2}\left(f(x)\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(f(x)\right)\right)$ and evaluating the first derivative to then take derivative again.

2. Find a pattern: (use induction if you feel confident about an old problem plus)

$$\frac{dy}{dx} = \frac{d}{dx}\left(xf(x)\right) = f(x) + xf'(x)$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(f(x) + xf'(x)\right) = 2f'(x) + xf''(x)$$
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(2f'(x) + xf''(x)\right) = 3f''(x) + xf'''(x)$$
$$\Rightarrow \frac{d^ny}{dx^n} = nf^{(n-1)}(x) + xf^{(n)}(x)$$

3.

$$y' = (g(2x))' = 2g'(2x)$$
$$y'' = \frac{d}{dx}\left(2g'(2x)\right) = 4g''(2x)$$
$$y''' = \frac{d}{dx}\left(4g''(2x)\right) = 8g'''(2x)$$
$$\Rightarrow y^{(n)} = 2^n \cdot g^{(n)}(2x)$$

4.

$$y' = \left(h(x^2)\right)' = 2xh'(x^2)$$
$$y'' = \frac{d}{dx}\left(2xh'(x^2)\right) = 2h'(x^2) + 4x^2h''(x^2)$$
$$y''' = \frac{d}{dx}\left(2h'(x^2) + 4x^2h''(x^2)\right) = 2h''(x^2)2x + 8xh''(x^2) + h'''(x^2)8x^3 = 12xh''(x^2) + 8x^3h'''(x^2)$$
$$y^{(4)} = \frac{d}{dx}\left(12xh''(x^2) + 8x^3h'''(x^2)\right) = 12h''(x^2) + 24x^2h'''(x^2) + 24x^2h'''(x^2) + 16x^4h^{(4)}(x^2)$$

5.

$$\frac{dy}{dx} = \frac{2x}{1-3y^2}$$
$$\frac{d^2y}{dx^2} = \frac{2 - 6y^2 + 12xy \cdot \frac{dy}{dx}}{(1-3y^2)^2}$$
$$\frac{d^3y}{dx^3} =$$
$$= \frac{\left(-12y \cdot \frac{dy}{dx} + 12y \cdot \frac{dy}{dx} + 12x \cdot \left(\frac{dy}{dx}\right)^2 + 12xy \cdot \frac{d^2y}{dx^2}\right)(1-3y^2)^2 - 2(1-3y^2)\left(-6y \cdot \frac{dy}{dx}\right)\left(2 - 6y^2 + 12xy \cdot \frac{dy}{dx}\right)}{(1-3y^2)^4}$$
$$= \frac{\left(12x\left(\frac{dy}{dx}\right)^2 + 12xy \cdot \frac{d^2y}{dx^2}\right)(1-3y^2)^2 + 12y(1-3y^2)\left(\frac{dy}{dx}\right)\left(2 - 6y^2 + 12xy \cdot \frac{dy}{dx}\right)}{(1-3y^2)^4}$$