Antiderivatives and Solving Differential Equations

Goal:
- Can find the antiderivative of polynomials, radicals, exponential, and simple rational functions.
- Can use antiderivatives to solve differential equations and use initial values to find the exact solution

Terminology:
- Antiderivative and $\int$
- Differential equation
- Initial value

Discussion question: What does the following symbol mean and do?

$$\frac{d}{dx}$$

It takes the derivative of functions. It is an **operator** like $\times, \div, +, -, \sqrt{(\ )}, \ln(\ )$ and so on.

Because it's an operator it should have a reverse operation

$$\frac{d}{dx}\big(f(x)\big) \mapsto f'(x)$$
$$anti - \frac{d}{dx}\big(f'(x)\big) \mapsto f(x)$$

This antiderivative is called an integral (more on this in the last unit)

$$\int f'(x) \mapsto f(x)$$

From the discussion it's natural to want to go backwards and solve the following problem. Given a known $f(x)$, determine $F(x)$ such that

$$\frac{d}{dx}\underbrace{\boxed{\ ?\ }}_{F(x)} = \underbrace{f(x)}_{\text{known}}$$

Example: Determine $F(x)$ such that

$$\frac{d}{dx}F(x) = \sqrt{x} + 3x^4$$

Think

$$\frac{d}{dx}\boxed{\ ?\ } = \sqrt{x} + 3x^4$$

$$\Rightarrow \frac{d}{dx}\boxed{Ax^{\frac{3}{2}} + 3 \cdot Bx^5} = x^{\frac{1}{2}} + 3x^4$$

This would get the right powers, but remember the powers come down so we need to fix coefficients.

$$\Rightarrow \frac{d}{dx}\boxed{\frac{2}{3}x^{\frac{3}{2}} + 3 \cdot \frac{1}{5}x^5 + C} = x^{\frac{1}{2}} + 3x^4$$

Now when the powers come down, we can see taking the derivative gives us the right function

** Remember that this is reversible so check!

Example: Determine $F(x)$ if

$$F'(x) = \frac{1}{x-3} + e^{-2x}$$

$$\frac{d}{dx}\boxed{\;?\;} = \frac{1}{x-3} + e^{-2x}$$

$$\Rightarrow \frac{d}{dx}\boxed{A \cdot \ln|x-3| + B \cdot e^{-2x}} = \frac{1}{x-3} + e^{-2x}$$

This would get us close, but we need to remember chain rule if we took the derivative.

$$\Rightarrow \frac{d}{dx}\boxed{1 \cdot \ln|x-3| - \frac{1}{2} \cdot e^{-2x}} = \frac{1}{x-3} + e^{-2x}$$

$$\Rightarrow F(x) = \ln|x-3| - \frac{e^{-2x}}{2} + C$$

Practice: Determine $F(x)$ such that

$$\frac{d}{dx}F(x) = \sqrt[3]{1-5x} + x^{-\frac{3}{2}}$$

$$\frac{d}{dx}\boxed{\;?\;} = (1-5x)^{\frac{1}{3}} + x^{-\frac{3}{2}}$$

$$\Rightarrow \frac{d}{dx}\boxed{A \cdot (1-5x)^{\frac{4}{3}} + B \cdot x^{-\frac{1}{2}}} = (1-5x)^{\frac{1}{3}} + x^{-\frac{3}{2}}$$

Again, we need to fix coefficients after doing power rule and chain rule

$$\Rightarrow \frac{d}{dx}\boxed{\frac{3}{4} \cdot \left(-\frac{1}{5}\right) \cdot (1-5x)^{\frac{4}{3}} - 2 \cdot x^{-\frac{1}{2}}} = (1-5x)^{\frac{1}{3}} + x^{-\frac{3}{2}}$$

$$\Rightarrow F(x) = \frac{-3}{20}(1-5x)^{\frac{4}{3}} - \frac{2}{\sqrt{x}} + C$$

Practice: Determine $F(x)$ such that

$$F'(x) = \frac{1}{x-4} + \frac{5}{1-2x} + 2^x$$

$$\frac{d}{dx}\boxed{\;?\;} = \frac{1}{x-4} + \frac{5}{1-2x} + 2^x$$

$$\Rightarrow \frac{d}{dx}\boxed{A \cdot \ln|x-4| + 5 \cdot B \cdot \ln|1-2x| + C \cdot 2^x} = \frac{1}{x-4} + \frac{5}{1-2x} + 2^x$$

$$\Rightarrow \frac{d}{dx}\boxed{1 \cdot \ln|x-4| + 5 \cdot \left(-\frac{1}{2}\right) \cdot \ln|1-2x| + \frac{1}{\ln 2} \cdot 2^x} = \frac{1}{x-4} + \frac{5}{1-2x} + 2^x$$

$$\Rightarrow F(x) = \ln|x-4| - \frac{5}{2}\ln|1-2x| + \frac{2^x}{\ln 2} + C$$

Common antiderivatives we need to memorize

$g(x)$	$g'(x) = \frac{d}{dx}g(x)$		$f(x)$	$F(x) = \int f(x)$
x^n	nx^{n-1}		$x^n + 0$	$\frac{1}{n+1} \cdot x^{n+1} + C$
e^x	e^x		$e^x + 0$	$e^x + C$
$\ln x$	$\frac{1}{x}$		$\frac{1}{x} + 0$	$\ln\|x\| + C$ * Note the absolute value
C	0		0	C

Since there is a constant C in every general antiderivative, when we solve differential equations we have solved them up to some vertical shift factor. We need to know some value of the function to find the exact solution. Hence, an **initial value** to **initialize** the solution.

Example: Solve for y given $y(0) = 2$

$$\frac{dy}{dx} = e^x + \frac{2}{x-1}$$

$$\Rightarrow y = e^x + 2\ln|x-1| + C$$

$$\Rightarrow 2 = e^0 + 2\ln|0-1| + C = 1 + C \Rightarrow C = 1$$

$$y = e^x + 2\ln|x-1| + 1$$

Example: Solve for V given $V(1) = 1$

$$\frac{dV}{dt} = \sqrt{1+3t} - 1$$

$$\Rightarrow V = \frac{2}{3} \cdot \frac{1}{3}(1+3t)^{\frac{3}{2}} - t + C$$

$$\Rightarrow 1 = \frac{2}{9}(4)^{\frac{3}{2}} - 1 + C \Rightarrow C = 2 - \frac{16}{9} = \frac{2}{9}$$

$$V = \frac{2}{9}\sqrt{(1+3t)^3} - t + \frac{2}{9}$$

Example: If F is tangent to the line $y = 3x$ and $F'(x) = x^2 - 1$ then determine F

$F'(x)$ is the slope of F and we know that the slope is 3 at the point of tangency. We can solve for x

$$x^2 - 1 = 3 \Rightarrow x = \pm 2$$

Therefore, the curve F is tangent to $y = 3x$ at $x = \pm 2$. We can find F up to a constant C

$$F(x) = \frac{x^3}{3} - x + C$$

And we know $F(x) = 3x$ at $x = \pm 2$ (since it's tangent at that point), thus

$$\frac{x^3}{3} - x + C = 3x$$

$$\Rightarrow C = 4x - \frac{x^3}{3}$$

$$C = \pm\left(8 - \frac{8}{3}\right) = \pm\frac{16}{3}$$

$$F(x) = \frac{x^3}{3} - x \pm \frac{16}{3}$$

Practice: Determine G given $G(-1) = 1$ and

$$G'(x) = 2x + \frac{1}{3-x}$$

$$\Rightarrow G(x) = x^2 - \ln|3-x| + C$$

$$\Rightarrow 1 = 1 - \ln 4 + C \Rightarrow C = \ln 4$$

$$\Rightarrow G(x) = x^2 - \ln|3-x| + \ln 4$$

Practice: If the function H is tangent to the curve $x + y = 1$ and $H'(x) = -e^{-2x}$ then what is H?

The slope of H is H' and we want it to be -1

$$-e^{-2x} = -1 \Rightarrow x = 0$$

Find H up to a constant C

$$H(x) = \frac{1}{2}e^{-2x} + C$$

And set it equal to the line $y = 1 - x$ at $x = 0$

$$H(0) = 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

$$H(x) = \frac{1}{2}e^{-2x} + \frac{1}{2}$$

Practice Problems: 9.1: # 1-3 & 5 (do what you need), 6
9.2: # 1, 2abd, 3 & 4bcd (do what you need), 5

9.1 # 7acd; 9.2 # 7