Steady States and Motion

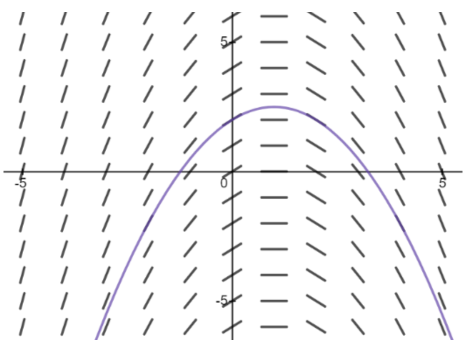
Goal:
- Can identify the steady states of a differential equation.
- Can use the steady states to predict behaviour of a particle in motion.

Terminology:
- Steady state
- Stable, unstable

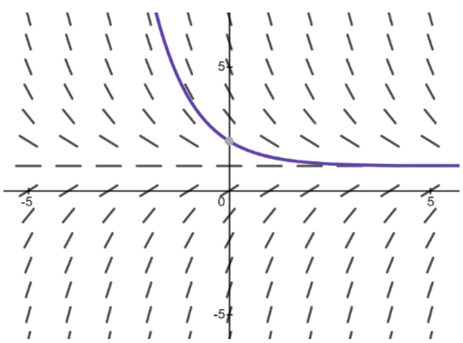
Discussion question: Three particles are moving along the y-axis. There vertical position are a, b, and c and their respective velocities are:

$$\frac{da}{dt} = 1 - t \qquad \frac{db}{dt} = 1 - b \qquad \frac{dc}{dt} = (1 - t) \cdot (1 - c)$$

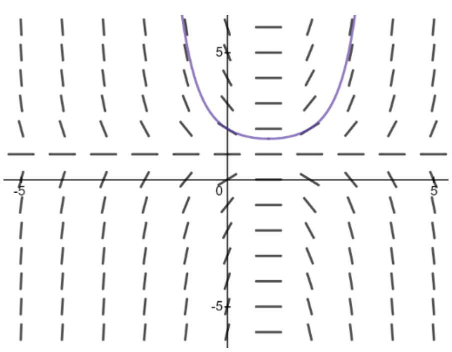
Every particle is at $y = 2$ when $t = 0$, that is $a(0) = b(0) = c(0) = 2$. How does motion differ for each particle?



Particle a will fall down the y-axis moving to $-\infty$. Although it does stop moving at $t = 1$, it will start to fall down immediately after. As $t \to \infty$ we have $\frac{da}{dt} \to -\infty$



Particle b will settle down at $b = 1$ (we call this a **stable steady state**). As $t \to \infty$ we have $\frac{db}{dt} \to 0$ as it is drawn toward $b = 1$



Particle c will shoot up the y-axis moving to ∞ (as it moves away from $c = 1$ which is an **unstable steady state** as if $c = 1$ then $\frac{dc}{dt} = 0$ no matter what t is). As $t \to \infty$ we have that $\frac{dc}{dt} \to \pm\infty$ depending on the sign of $1 - c$

A differential of the form

$$\frac{dy}{dt} = f(t)$$

Likely has NO steady states even though $\frac{dy}{dt}$ may be 0 at some time t. A steady state is when $\frac{dy}{dt} \to 0$ as $t \to \infty$

So $f(t)$ would need a horizontal asymptote but it is still going to depend on the initial condition. ALSO, it is not even enough we have a horizontal asymptote

Example: Consider the functions

$$\frac{dy}{dt} = t \qquad \frac{dy}{dt} = \frac{1}{t} \qquad \frac{dy}{dt} = \frac{1}{t^2}$$

We can solve for y in all of these cases

$$\frac{d}{dt}[???] = t \Rightarrow y = \frac{t^2}{2} + C$$

Here $\frac{dy}{dt}$ has no horizontal asymptote and the solution is a parabola so it $y \to \infty$ as $t \to \infty$

$$\frac{d}{dt}[???] = \frac{1}{t} \Rightarrow y = \ln|t| + C$$

Here $\frac{dy}{dt}$ has a horizontal asymptote of 0 which suggest as steady state, but the solution is a log so still we have $y \to \infty$ as $t \to \infty$

$$\frac{d}{dt}[???] = \frac{1}{t^2} \Rightarrow y = -\frac{1}{t} + C$$

In the last case $\frac{dy}{dt}$ has a horizontal asymptote of 0 and the solution is $\frac{1}{t}$ so there is a steady state of $y \to C$ as $t \to \infty$

In general, we will not be looking much at differentials where $\frac{dy}{dt} = f(t)$ for the remainder of the unit.

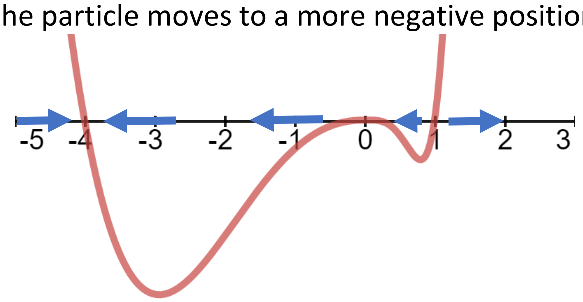
Example: Determine the steady states and their stability of

$$\frac{dy}{dt} = y^2(y-1)(y+4)$$

We look for where $\frac{dy}{dt} = 0$ this is the zeros of the polynomial given and at

$$y = 0, 1, -4$$

To see the behaviour around these steady states we can sketch the differential equation as a function of y. Using the sign of y' we can see where the particle would move. If $y' > 0$ then the particle moves to a more positive position and if $y' < 0$ the particle moves to a more negative position.



We have that -4 is stable, 0 is semi-stable and 1 is unstable

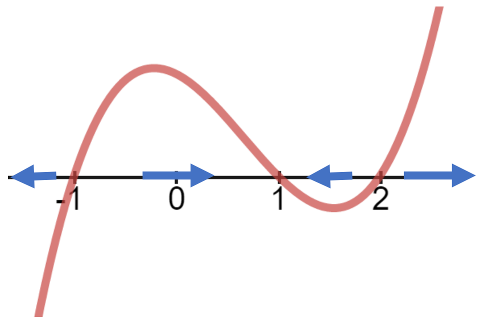
Practice: Determine the steady states and stability of

$$\frac{dP}{dt} = (P^2 - 1)(2 - P)$$

We look for where $\frac{dP}{dt} = 0$ which is when

$$P = -1, 1, 2$$

We can sketch $\frac{dP}{dt}$ as a function of P and then



We have that 1 is stable, and -1 and 2 are both unstable

Practice: Determine the steady states and stability of

$$\frac{da}{db} = (1 - b)(2a + 4)$$

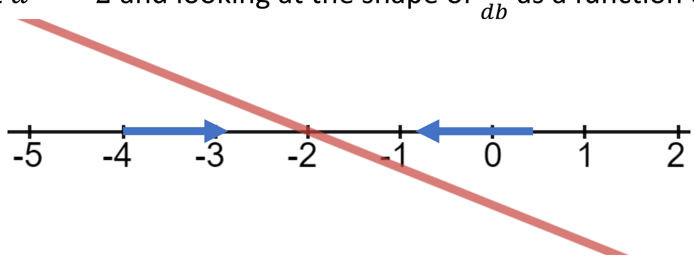
When we are looking for steady states in

$$\frac{dy}{dt} = f(t, y)$$

We want $t \to \infty$ and to find the values of y that make $\frac{dy}{dt} = 0$. In this case we want to find the values of a that make $\frac{da}{db} = 0$ and let $b \to \infty$. So, we essentially have

$$\frac{da}{db} = -(2a + 4)$$

Which has a steady state at $a = -2$ and looking at the shape of $\frac{da}{db}$ as a function of a we get



Showing that $a = -2$ is stable.

Practice Problems: Steady state practice problems