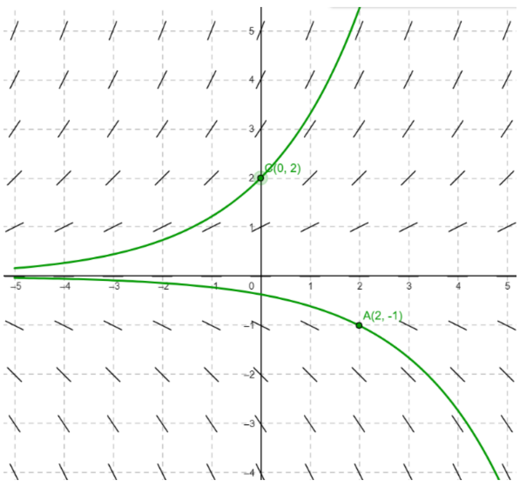
Natural Growth: Part 1

Goal: • Can solve the initial value differential equation $\frac{dy}{dt} = ky,\ y(0) = y_0$ • Can use steady states to interpret natural growth and create a differential equation that matches it.	
Terminology: • Natural Growth	

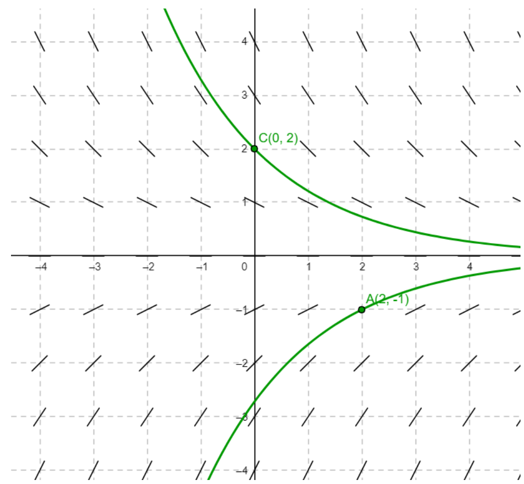
Discussion question: What does the function $y(t)$ look like if

$$\frac{dy}{dt} = ky$$

Consider both cases where the constant $k > 0$ and $k < 0$.



Unstable steady state at $y = 0$ when $k > 0$



Stable steady state at $y = 0$ when $k < 0$

In either case the solution curve will look like: *an exponential curve*

And we can check that a valid solution to the differential equation is

$$y(t) = Ce^{kt}$$

Note that the constant C, appears as a vertical *explansion* instead of a vertical translation. This is because

There must be a steady state at $y = 0$ and therefore a horizontal asymtote at $y = 0$. The curve cannot shift up or down.

Example: Solve the differential equation

$$\frac{dy}{dt} = 0.05y, \qquad y(0) = 2$$

The differential $\frac{dy}{dt} = kt$ has the solution $y = Ce^{kt}$

$$\Rightarrow y = Ce^{0.05t}$$
$$2 = Ce^0 = C$$

$$y = 2e^{0.05t}$$

In this differential equation k determines how fast the curve grows but the overall shape stays the same. Since we have

$$\frac{dy}{dt} = ky \Rightarrow k = \frac{1}{y} \cdot \frac{dy}{dt}$$

And so k is the growth rate per unit population (per capita)

Example: Your body metabolizes sugar at a per capita rate of 240% per hour. Write the differential equation that describes the metabolic rate and solve it if the initial mass ingested is 200 g

The amount of sugar, s, will go to 0 as $t \to \infty$. Therefore we have $s = 0$ is a stable steady state $(k < 0)$

$$\frac{ds}{dt} = -2.4s, \qquad s(0) = 200$$

$$s = Ce^{-2.4t}$$
$$200 = Ce^{0} = C$$

$$\Rightarrow s(t) = 200 \cdot e^{-2.4t}$$

Practice: Solve the differential equations and state the per capita growth rate.

$$\frac{dN}{dt} = -0.3N, \qquad N(0) = 40$$

We know the form of the solution so we get

$$N = Ce^{-0.3t}$$
$$40 = Ce^{0} = C$$

$$\Rightarrow N(t) = 40 \cdot e^{-0.3t}$$

The per capita growth rate is 30% per unit time (decay as the steady state is stable)

Practice: An investment grows at a per capita rate of 8% per year. Write the differential equation that describes the rate the investment grows and solve it if the initial investment is $20,000.

The steady state of amount of money, A, is $A = \$0$ and this is unstable since having any more money will gain more. $(k > 0)$

$$\frac{dA}{dt} = +0.08A$$
$$\Rightarrow A(t) = Ce^{0.08t}$$

$$20000 = Ce^{0} = C$$

$$A(t) = 20000 \cdot e^{0.08t}$$

Typically when we build a differential equation we need set up the dfferential equation by identifying the steady state and the stability. And then we need to measure the growth rate by actually making measurements (one measurement for the growth rate and one measurement for our initial condition).

Example: A faulty battery loses charge at a rate proportional to the current charge. After 15 minutes the battery has just 1/5th of the initial charge. What is the differential equation that describes the charge and what is the exact solution to the differential equation?

The steady state of battery charge, Q, is $Q = 0$ and this is stable since a battery will lose charge naturally. $(k < 0)$

$$\frac{dQ}{dt} = kQ, \qquad Q(0) = q_0, \qquad Q(15) = \frac{1}{5}q_0$$
$$\Rightarrow Q(t) = Ce^{kt}$$

$$q_0 = Ce^0 = C$$
$$\Rightarrow Q(t) = q_0 e^{kt}$$

$$\frac{1}{5}q_0 = q_0 \cdot e^{k(15)}$$
$$\frac{1}{5} = e^{15k} \Rightarrow \ln\left(\frac{1}{5}\right) = 15k \Rightarrow k = -\frac{1}{15}\ln 5 = -0.107 \text{ min}^{-1}$$

As expected, $k < 0$ and it appears the battery loses charge at a rate of about 11% per minute

$$Q(t) = q_0 e^{-0.107t}$$

Practice: Atmospheric pressure decreases at a rate proportional to the current pressure as height changes. If one day at a certain location the atmospheric pressures are 760 and 675 torr (unit for pressure) at sea level and at 1000 meters above sea level, respectively, find the value of the atmospheric pressure at 600 meters above sea level.

The steady state of pressure is $P = 0$ and is stable as $P \to 0$ as $h \to \infty$ $(k < 0)$

$$\frac{dP}{dh} = kP, \qquad P(0) = 760, \qquad P(1) = 600$$
$$\Rightarrow P(h) = Ce^{kh}$$

$$760 = Ce^0 = C$$
$$\Rightarrow P(h) = 760e^{kh}$$

$$600 = 760 \cdot e^{k(1)}$$
$$\frac{600}{760} = e^k \Rightarrow \ln\left(\frac{60}{76}\right) = k = -0.236 \text{ m}^{-1}$$

Again, $k < 0$ and it appears the pressure is decreasing very slowly at 24% per kilometer.

$$P(h) = 760e^{-0.236h}$$
$$P(0.6) = 660 \text{ torr}$$

Practice Problems: 9.4 # 1-3
Basic Growth and Decay Practice Problems.