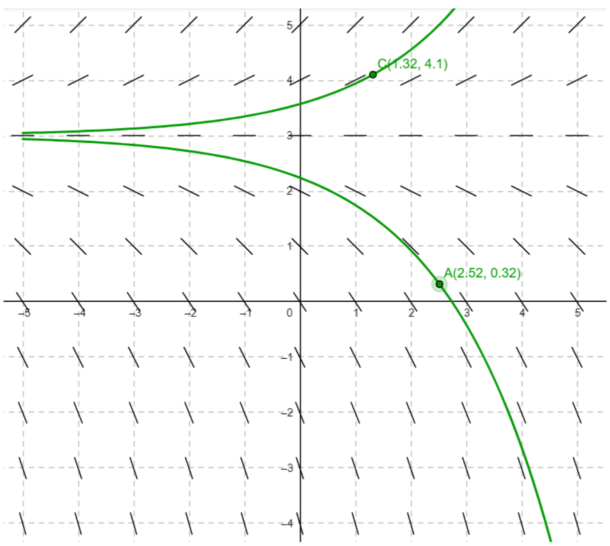
Natural Growth: Part 2

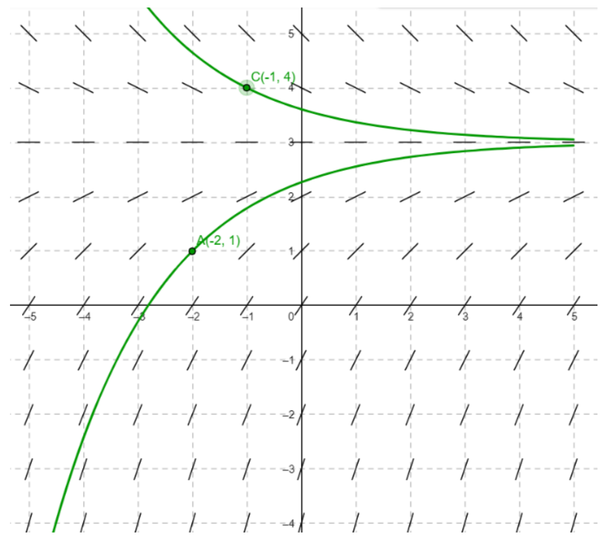
Goal: • Can solve the initial value differential equation $\frac{dy}{dt} = k(y - M),\ y(0) = y_0$ using steady states. • Can use steady states to interpret natural growth and create a differential equation that matches it.
Terminology: • None.

Discussion question: What does the function $y(t)$ look like if

$$\frac{dy}{dt} = k(y - M)$$

Again, consider $k > 0$ and $k < 0$. How is this different than the differntial we looked at last class?





When $k > 0$ we have a nonzero unstable steady state and when $k < 0$ we have a stable srteady state at some higher point. This looks like an exponential just shifted up or down.

How can we solve this differential equation?? The answer is to use steady states and that $\frac{d}{dt}(y + c) = \frac{dy}{dt}$ when c is constant

Consider the differential equation

$$\frac{dy}{dt} = k(y - M)$$

Shift the steady state to be $y = 0$ (shift everything down M units) and then let $y - M = \Gamma$ but note that since $\frac{dM}{dt} = 0$

$$\frac{dy}{dt} = \frac{d\Gamma}{dt}$$

$$\Rightarrow \frac{dy}{dt} = k \cdot (\Gamma) = \frac{d\Gamma}{dt}$$

The highlighted equation is easily solved,

$$\Gamma = Ce^{kt} = y - M$$

$$\Rightarrow y = Ce^{kt} + M$$

And we change back to y by shifting up M units.

Example: Determine the solution to the differential

$$\frac{dy}{dt} = 2 - 6y, \qquad y(0) = 4$$

We see the steady state is $y = \frac{1}{3}$, so we let $y - \frac{1}{3} = \Gamma$ so Γ has a steady state of 0 but they have the same slope.

$$\Rightarrow \frac{dy}{dt} = 2 - 6\left(\Gamma + \frac{1}{3}\right) = -6\Gamma = \frac{d\Gamma}{dt}$$

Solve for Γ

$$\Gamma = Ce^{-6t}$$

Now if $y = 4$ when $t = 0$ that means $\Gamma = 4 - \frac{1}{3} = \frac{11}{3}$

$$\Rightarrow \Gamma = \frac{11}{3}e^{-6t}$$

$$\Rightarrow y = \frac{11}{3}e^{-6t} + \frac{1}{3}$$

Practice: Solve the differential

$$\frac{dy}{dt} = \frac{10y + 15}{30}, \qquad y(0) = 0$$

Find that the steady state is $y = -\frac{3}{2}$, so we let $y + \frac{3}{2} = \Gamma$ so Γ has a steady state of 0 but they have the same slope.

$$\Rightarrow \frac{dy}{dt} = \frac{10\left(\Gamma - \frac{3}{2}\right) + 15}{30} = \frac{10}{30}\Gamma = \frac{d\Gamma}{dt}$$

Solve for Γ

$$\Gamma = Ce^{\frac{1}{3}t}$$

Now if $y = 0$ when $t = 0$ that means $\Gamma = \frac{3}{2}$

$$\Rightarrow \Gamma = \frac{3}{2}e^{\frac{1}{3}t}$$

$$\Rightarrow y = \frac{3}{2}e^{\frac{1}{3}t} - \frac{3}{2}$$

When we are presented with a situation with an application that we want to solve we need to identify three things in the problem

- The steady state and the stability. This will give us most of the differential equation and prep us for the sign of the per capita growth rate. This will be given from the contex of the problem or seen as a limit as $t \to \infty$. For temperature this will be the ambient room temperature, for a population this will be the carrying capacity, for collection this will be the amount needed, etc.

- The initial condition which we should typically set as $t = 0$ for ease of calculation. This is measured experimentally. Some measured value after $t = 0$. This will determine the rate of growth and will be measured experimentally.

Example: Consider someone who is trying to collect 248 Amazing Spiderman comics only through random chance. Then the rate they collect new comic books is proportional to the number of books they need to collect. 1 year after they collect their first comic, they have 37 unique comics. Detemine a differential equation for this problem and solve it to predict how many they will have collected after 15 years.

The (stable) steady state will be 248 since at that point there will be no new comics to find. In other words, as $t \to \infty$ the number of comics collected, n, will converge to 248.

$$\frac{dn}{dt} = k(n - 248), k < 0$$

Set $n - 248 = \Gamma$

$$\Rightarrow \frac{dn}{dt} = k\Gamma = \frac{d\Gamma}{dt}$$

Solve for Γ

$$\Gamma = Ce^{kt} = n - 248$$
$$\Rightarrow n = Ce^{kt} + 248$$

Solve for C by letting $t = 0$ and $n = 1$

$$\Rightarrow 1 = C + 248 \Rightarrow C = -247$$

Solve for k by letting $t = 1$ and $n = 37$

$$\Rightarrow 37 = -247e^{k} + 248 \Rightarrow k = \ln\left(\frac{211}{247}\right) = -0.158 \text{ year}^{-1}$$

$$n(t) = -247e^{-0.158t} + 248$$
$$n(15) = 225 \text{ comics}$$

Practice: An overweight person is beginning to exercise and diet with a plan designed to help them reach a target weight of 180 lbs. They will lose weight faster at the beginning of the program while they have more weight to lose. If they are currently 270 lbs and after 5 weeks they are 258 lbs, when will they be 200 lbs?

The (stable) steady state will be 180 since they will stay at the weight if their routine stays the same. In other words, as $t \to \infty$ their weight, $w \to 180$

$$\frac{dw}{dt} = k(w - 180), k < 0$$

Set $w - 180 = \Gamma$

$$\Rightarrow \frac{dw}{dt} = k\Gamma = \frac{d\Gamma}{dt}$$

Solve for Γ

$$\Gamma = Ce^{kt} = w - 180$$
$$\Rightarrow w = Ce^{kt} + 180$$

Solve for C by letting $t = 0$ and $w = 270$

$$\Rightarrow 270 = C + 180 \Rightarrow C = 90$$

Solve for k by letting $t = 5$ and $w = 258$

$$\Rightarrow 258 = 90e^{5k} + 180 \Rightarrow k = \frac{1}{5}\ln\left(\frac{78}{90}\right) = -0.0286 \text{ week}^{-1}$$

$$w(t) = 90e^{-0.0286t} + 180$$
$$200 = 90e^{-0.0286t} + 180 \Rightarrow t = 52.6 \text{ weeks}$$

Practice Problems: 9.4 # 4-9