Mixing

Goal:
- Can solve problems involving multiple rates of change using linear differential equations.
- Can use steady states to verify solutions and build equations

Terminology:
- Concentration

Discussion question: A population of deer reproduce at a rate where their population would grow at a per capita rate of 15% per year if there were no outside factors. The deer are hunted at a controlled rate of 1200 deer per year. What will happen to the population if there are initially 5000 deer? 10 000 deer?

The big idea is that there is some number of deer where the population growth will equal the amount being hunted and so the population will not change. This is out Steady State!

To find the steady state we have a population of deer, P, and the amount gained in a year will be $0.15P$ which we want to be 1200. Therefore, the steady state (unstable) is $P = \frac{1200}{0.15} = 8000$. If there are less than 8000 deer the population of deer will not be sustainable but if we have over 8000 deer the population will continue to grow.

$$\frac{dP}{dt} = k(P - 8000), \qquad k = 0.15$$

Let $\Gamma = P - 8000$ so $\frac{dP}{dt} = k\Gamma = \frac{d\Gamma}{dt}$ and we get $\Gamma = Ce^{0.15t} = P - 8000$

1. If we start with 5000 deer, then $C = -3000$ and as $t \to \infty$ then $P \to -\infty$ and the population will be extinct when $0 = -3000e^{0.15t} + 8000$ which is when $t = 6.5$ years

2. If we start with 10 000 deer, then $C = 2000$ and as $t \to \infty$ we have $P \to \infty$

When analyzing problems like this we can set our work up in one of two ways.

Method 1

Find the steady state and determine k from the context of the problem. This method requires we be able to determine the steady state from a holistic approach and that we have enough information to determine k

$$\frac{dP}{dt} = k(y - 8000)$$

Method 2

Set up the differential equation using the idea that total rate of change is just the sum of the smaller rates of change

$$\frac{dy}{dt} = \text{rate}_{\text{in}} - \text{rate}_{\text{out}}$$

For the above problem we would have

$$\frac{dP}{dt} = kP - 1200$$

Where $\frac{dP}{dt}$ has units of deer per year. Note that if you factor the k out you get $k\left(P - \frac{1200}{k}\right) = k(P - 8000)$ for $k = 0.15$

Example: A fish tank has a volume of 200 L of fresh water. You want to raise the concentration of salt slowly and begin dripping in a concentrated saline solution of 50 g/L at a rate of 4 L/hour. Water is removed from the tank at the same rate. Make a differential equation that models the situation.

Method 1: (Change in concentration) Note that the steady state of concentration will be 50 g/L since as $t \to \infty$ the water in the tank should be entirely full of the saline solution. Let $[s]$ be the concentration of salt in g/L

$$\frac{d[s]}{dt} = k([s] - 50)$$

We want $\frac{d[s]}{dt}$ to have units of g/L per hour so we need to make k in units per hour and the only thing we have is that water is being added at a rate of 4 L/hour. We want to make this a per capita growth so relate to the volume in the tank, $k = 4\frac{\cancel{\text{L}}}{\text{hour}} \cdot \frac{1}{200\ \cancel{\text{L}}} = 0.02\ \text{hour}^{-1}$

This can be solved as per normal, but it is not exactly intuitive why we should make k this way.

Method 1: (Change in mass) This is the same idea as above but rather than tracking the concentration, we just focus on the mass of salt in the tank (we could then find the concentration by dividing by 200 L). Since the final concentration would be 50 g/L and the volume is 200 L, the stable steady state should be 10 000 g. Let s be the mass of salt in the tank in kg

$$\frac{ds}{dt} = k(s - 10\ 000)$$

The value of k is found in the same way as before as it should if we replaced $[s] = \frac{s}{200}$ we get the same differential.

Method 2: (Change in mass) It doesn't make sense to talk about "concentration in per hour" since we can't add concentration in a measurable way as it is a calculated ratio not a physical measurement. What we can add per hour is mass of salt and volume of water.

Since we add salt at a concentration of 50 g/L and rate of 4 L/hour we are adding 200 g/hour. The salt is leaving with some concentration $\frac{s}{200}$ where s is the current amount in g and at the same rate of 4 L/hour. This means that salt leaves at a rate of $\frac{4s}{200} = \frac{s}{50}$ g/hour

$$\frac{ds}{dt} = 200 - \frac{s}{50}$$

Which is the exact same differential as above but just written in a different form. Note that here we did not need to solve for k but we don't immediately know our steady state.

Solving any of these differential equations with $\Gamma = s - 10000$ or $\Gamma = [s] - 50$ we get

$$\frac{ds}{dt} = 200 - \frac{\Gamma + 10000}{50} = -0.02 \cdot \Gamma \Rightarrow \Gamma = Ce^{-0.02t}$$

$$s = -10000e^{-0.02t} + 10000$$
$$[s] = -50e^{-0.02t} + 50$$

Practice: A 100 L tank of fresh water has fertilizer being pumped in with a concentration of 0.1 kg/L at a rate of 2 L/min. Water is leaving the tank at:

a. A rate of 2 L/min
b. A rate of 4 L/min

Write a differential equation for the amount of fertilizer in the tank for both parts and solve the equation for part "a", and predict the amount of fertilizer in the tank after 30 minutes.

A:
We can see that the steady state (stable) of concentration should be 0.1 kg/L and the rate it is adding is 2 L/min × 1/100L so $k = \frac{2}{100} = 0.02$

$$\frac{d[f]}{dt} = -0.02([f] - 0.1)$$

Let $\Gamma = [f] - 0.1$ and we get $\Gamma = Ce^{-0.02t} = [f] - 0.1$ with the knowledge that $[f] = 0$ when $t = 0$ so $C = -0.1$

$$\Rightarrow [f] = -0.1e^{-0.02t} + 0.1$$

$$\Rightarrow [f](30) = 0.045 \frac{\text{kg}}{\text{L}} \Rightarrow f = 4.5 \text{ kg}$$

Alternatively, we can make f the amount of fertilizer and that it is added at a rate of 0.2 kg/min and it leaves at a rate of $\frac{f}{100} \cdot 2$ kg/min

$$\frac{df}{dt} = 0.2 - \frac{f}{50}$$

Let $\Gamma = f - 10$ so

$$\frac{df}{dt} = 0.2 - \frac{\Gamma + 10}{50} = -0.02\Gamma = \frac{d\Gamma}{dt} \Rightarrow \Gamma = Ce^{-0.02t} = f - 10$$

$$f = -10e^{-0.02t} + 10$$
$$\Rightarrow f(30) = 4.5 \text{ kg}$$

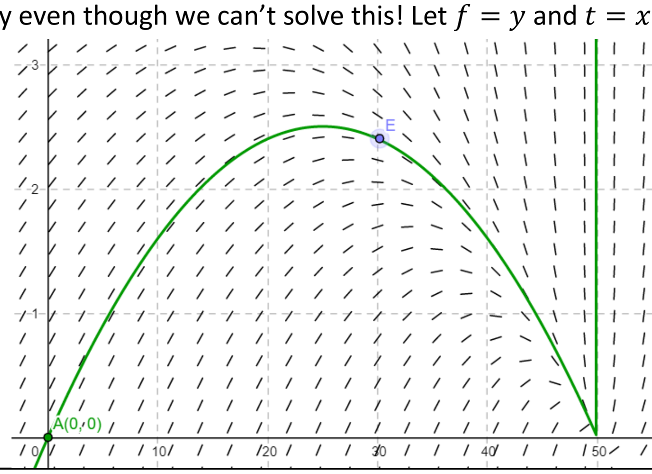
B:
Now we can't really see what a steady state would be since water is draining out of the tank faster than it is being replenished. Clearly, the tank will empty after 50 minutes and so $f(0) = f(50) = 0$ but it is not clear what the amount of fertilizer will be in between. We can make a differential equation since fertilizer is being added at a rate of 0.2 kg/min still and it leaves at a rate of

$$\frac{f \text{ (kg)}}{V(t) \text{ (L)}} \cdot 4 \text{ (L/min)}$$

Where $V(t) = 100 - 2t$ is the volume at time t (since it is not constant).

$$\Rightarrow \frac{df}{dt} = 0.2 - \frac{4f}{100 - 2t} = 0.2 - \frac{2f}{50 - t}$$

We can graph the slope field and guess $f(30)$ using technology even though we can't solve this! Let $f = y$ and $t = x$



Looking at the slope field it appears that after 30 minutes, the Amount of fertilizer is around 2.4 kg.

Also looking at the slope field it seems like a parabola is the solution to f. You can check that the good guess of

$$f = \frac{t(50 - t)}{250}$$

Is the actual solution to the differential equation

Practice: A TFSA earns 6% interest annually (per capita continuously). You deposit $2000 each month into the account and you made an initial invesment of $5000. Write a differential equation if

a. You withdrawl $0 each month
b. You withdrawl 1% of your account each month.
c. You add $10 each month than the month before. So the first month you add $2000, the next $2010, the next $2020 and so on.

Write a differential equation for the above and make a solution where possible. Predict the amount of money in the account after 25 years.

A:
We need to adjust our units of time from years to months since the $2000 will gain interest. Instead of 6%/year we will get 0.5%/month. Let A be the amount of money in the account (in thousands). Since it is not clear what the steady state would be we need to look at the rate money goes in and the rate it leaves the account.

$$\frac{dA}{dt} = 0.005A + 2$$

So the steady state would be $-$\$400K (doesn't really make sense but let's shift it to 0). Let $\Gamma = A + 400$

$$\frac{dA}{dt} = 0.005(\Gamma - 400) + 2 = 0.005\Gamma \Rightarrow \Gamma = Ce^{0.005t} = A + 400$$

$$A(0) = 5 \Rightarrow C = 405$$
$$A = 405e^{0.005t} - 400$$
$$A(25 \cdot 12) = \$1415 \text{ K}$$

B:
Now we just want to add a monthly withdrawal of $-0.001A$

$$\frac{dA}{dt} = 0.005A + 2 - 0.001A = 0.004A + 2$$

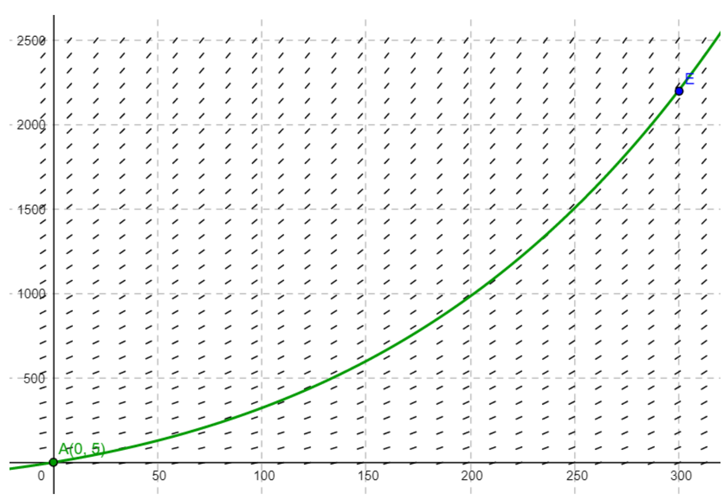
This changes our steady state slightly $\Gamma = A - 500$

$$\Rightarrow \Gamma = Ce^{0.004t} = A + 500$$
$$A = 505e^{0.004t} - 500$$
$$A(25 \cdot 12) = \$1177 \text{ K}$$

C:
Now we add $2 + 0.01t$ in thousand dollars each month.

$$\frac{dA}{dt} = 0.005A + 2 + 0.01t$$

We can't solve this but we can graph it! After 300 months it looks like there would be around 2.2 million in the account.



Note that if you plot the slope field and solution curve as a function to years that the predicted amount drops to 1.5 million

$$\frac{dA}{dt} = 0.06A + 24 + 0.12t$$

Practice Problems: 9.5 # 1-4 (do what you need) 5, 6, 8
Mixing Practice Problems.