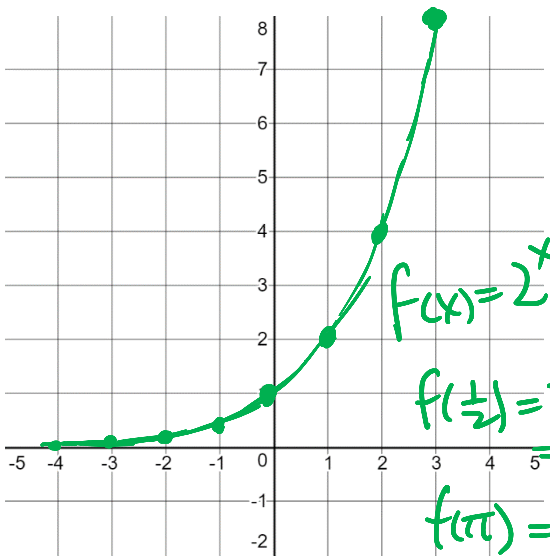
Exponential Functions

KNOW	**DO**	**UNDERSTAND**
Can recognize an exponential as increasing or decreasing from given points. Can identify the asymptote from the graph.	Can build equations for exponentials given two points and the asymptote.	*Function Characteristics*: The base is the rate of growth/decay, and the horizontal stretch is the length of one growth period. The vertical stretch is how far from the asymptote you start and the vertical shift is the asymptote.

Vocab & Notation

- Exponential growth/decay
- Euler's number, e

Consider the geometric sequence $(2^k)_{k\in\mathbb{Z}}$. Plot it on the graph below.



$(2^k)_{k\in\mathbb{Z}} = (\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots)$

from geometric sequence $\rightarrow$ exponential function

$f(x) = 2^x$

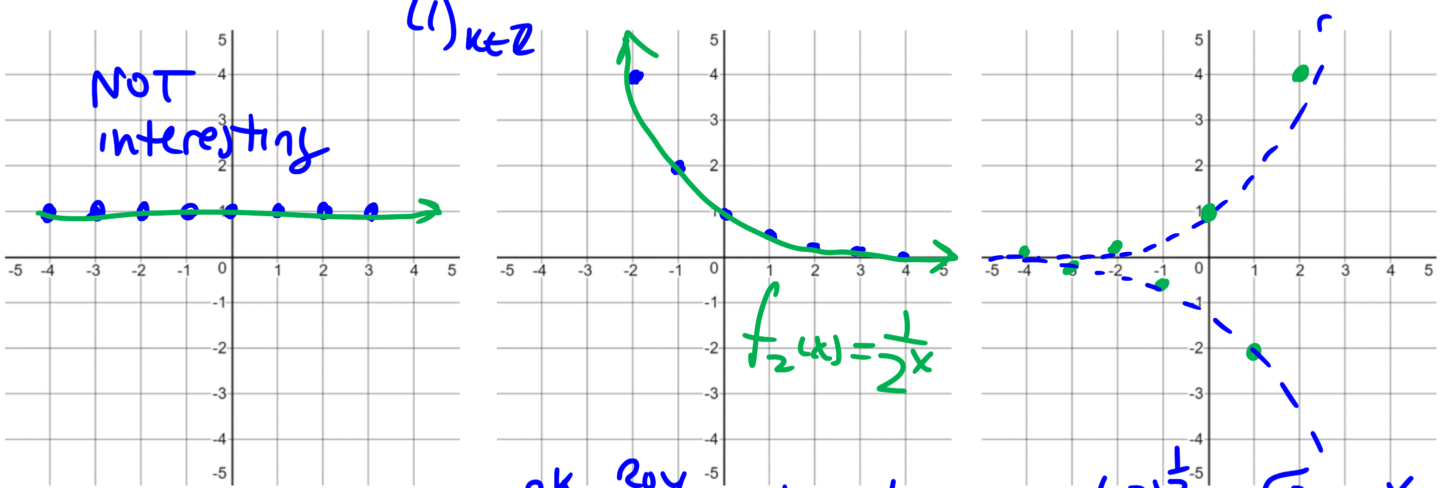
$f(\frac{1}{2}) = 2^{1/2} = \sqrt{2}$ ✓

Domain $\mathbb{R}$

$f(\pi) = 2^{\pi}$?? ✓ But calculate OK

Do the same for the sequences $(1^k)_{k\in\mathbb{Z}}$; $\left(\frac{1}{2^k}\right)_{k\in\mathbb{Z}}$; and $((-2)^k)_{k\in\mathbb{Z}}$ and graph them below.

$(1)_{k\in\mathbb{Z}}$



2^k $\underline{2oy}$ $2^{-k} = \frac{1}{2^k}$

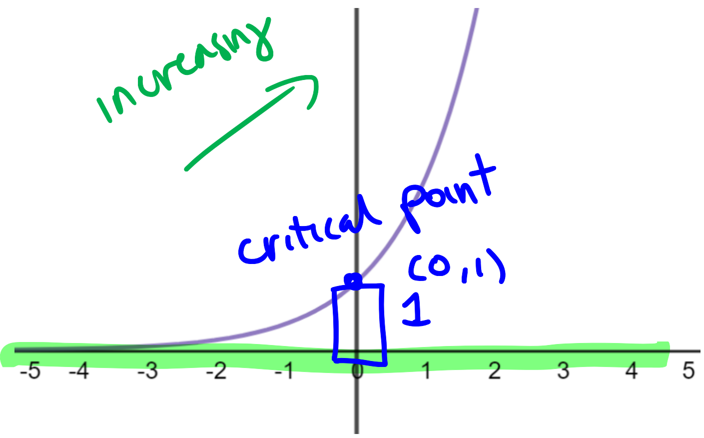
$(-2)^{\frac{1}{2}} = \sqrt{-2}$ ✗

$(-2)^{1/3} = \sqrt[3]{-2}$ ✓

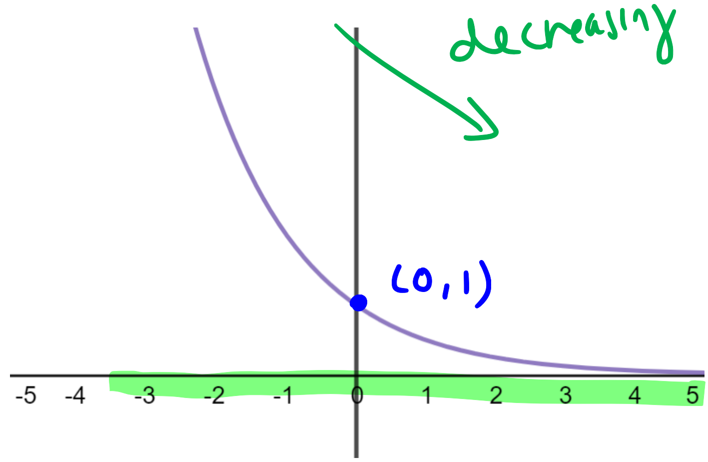
$(-2)^{\pi}$, $\pi \notin$ Domain

☆ only want bases that are >0 and $\neq 1$

The general graph form $y = b^x$ for $b > 1$



If $b \in (0,1)$, then



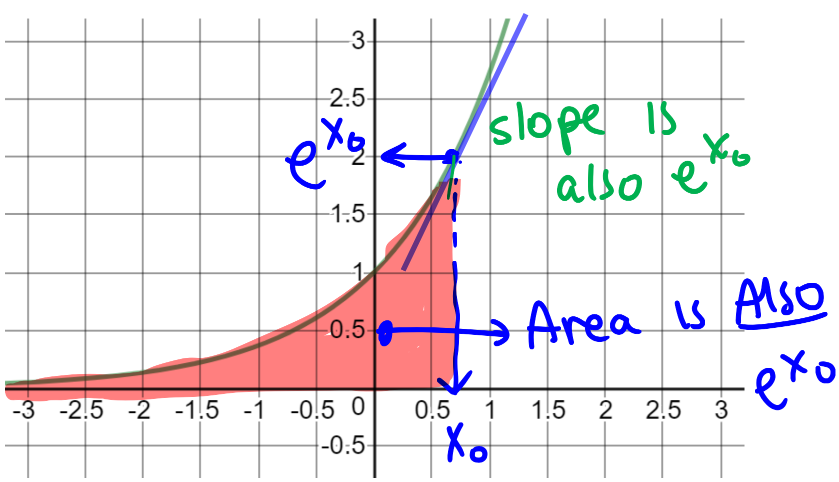
☆ all exponential functions pass thru (0,1) because $b^0 = 1$

☆ horizontal asymptote @ $y=0$ (on one end)

Arguably the most important exponential function uses **Euler's Number:**

$$f(x) = (2.71828\ldots)^x = e^x$$

THE exponential function



the area, slope + value are all e^x

$e^{ix} = \cos x + i \sin x$

$$\boxed{e^{i\pi} + 1 = 0}$$

And the important thing is that we can change from one base to another.

$2 = e^{\boxed{0.7}}$

$$f(x) = 2^x = (e^{0.7})^x = \underline{e^{0.7x}}$$

We'll talk more about this next tomorrow on finding that power for e, but right now just be comfortable that we can do this at least through guess and check and when you start doing calculus you will only want to work exclusively with base e.

For now, let's look to see how any exponential function could be defined by an arbitrary base.

$4 = e^{\boxed{1.4}}$ $12 = e^{\boxed{2.49}}$

Consider the specific case where a population doubles every year.

Population	1	2^1	2^2	2^3	2^4	. . .	2^t
Time	0	1	2	3	4	. . .	t

geometric sequence EZ

And now, the general case where a population grows at a rate r and it takes T years to grow that much or decay to that little.

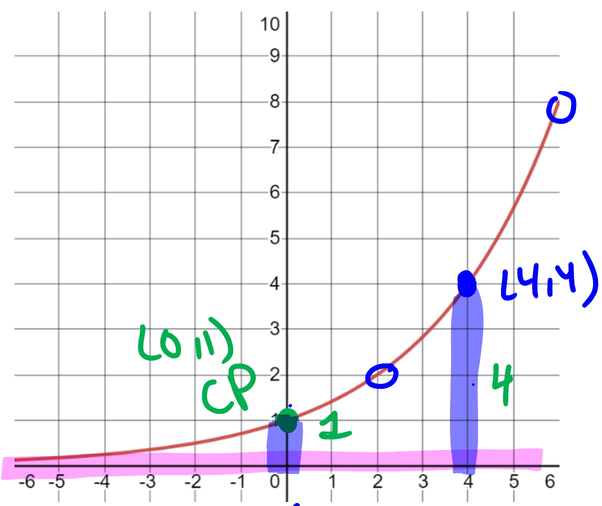
$\times r$

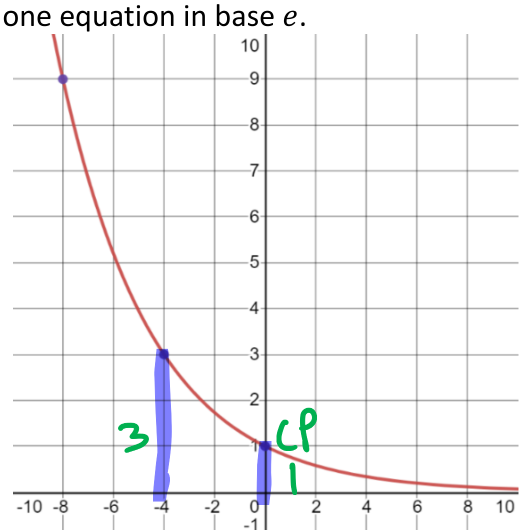
Population	1	r	r^2	r^3	r^4	. . .	r^n
Time	0	T	$2T$	$3T$	$4T$	. . .	$nT = t$

$n = \frac{t}{T}$

$P(t) = r^n = r^{\frac{t}{T}} \Rightarrow f(x) = r^{\frac{x}{T}}$ ← horizontal stretch by T

Example: Determine two equations for the following graphs, and one equation in base e.





$r = 4$ $T = 4$

$f(x) = 4^{\frac{x}{4}}$ $4 \sim e^{1.4}$

$\sim (e^{1.4})^{\frac{x}{4}} = e^{0.35x}$

$T = -4$ $r = 3$

$f(x) = 3^{-\frac{x}{4}} = \left(\frac{1}{3}\right)^{\frac{x}{4}}$

$\sim (e^{1.1})^{-x/4} = e^{-0.275x}$

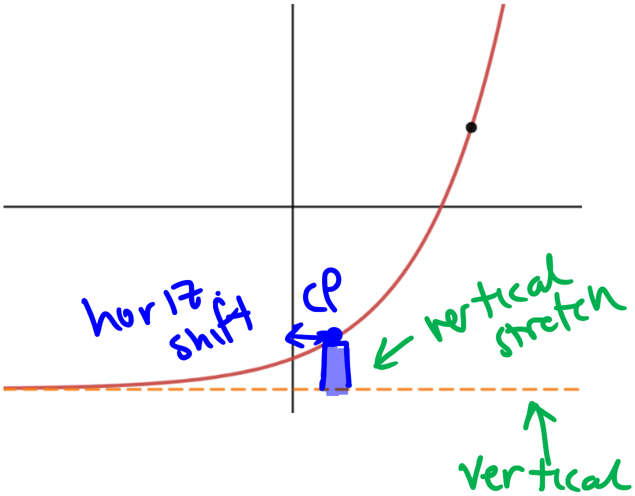
$3 \sim e^{1.1}$

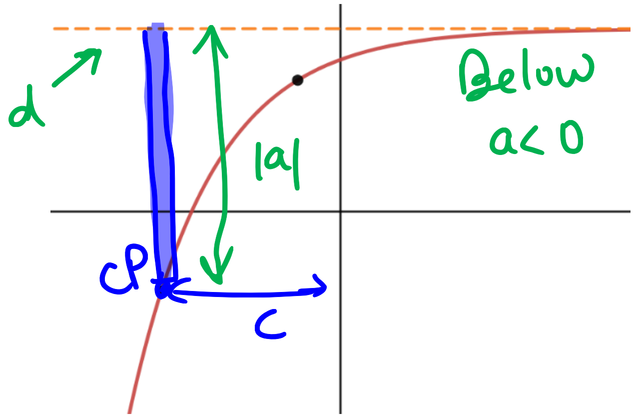
We've looked at doing a horizontal stretch, now we want to add the three other transformations to the exponential.

$$f(x) = a \cdot b^{\frac{x-c}{T}} + d = a \cdot e^{k(x-c)} + d$$

After a transformation we will be in one of four situations:

- Increasing or decreasing
- Approaching the asymptote or approaching infinity



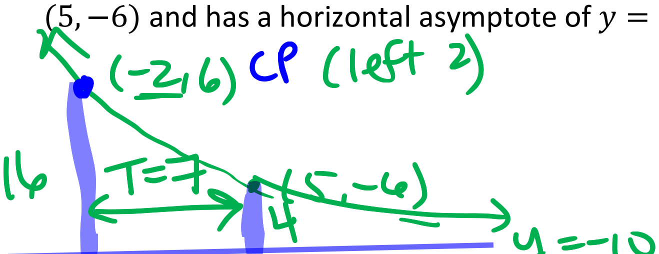


☆ asymptote is only moved by vertical shift

☆ pick critical point, we can find horiz. shift from here

☆ vertical stretch is distance from critical point to asymptote

Example: Determine an equation in base e for the exponential that passes through the points $(-2, 6)$ and $(5, -6)$ and has a horizontal asymptote of $y = -10$.

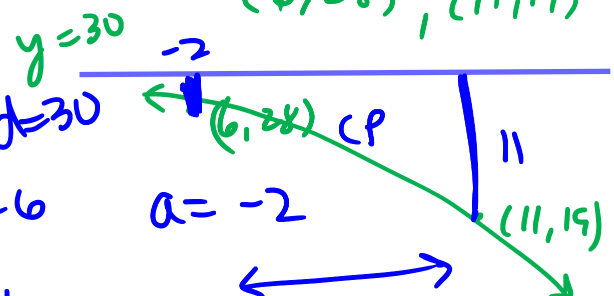


$d=-10$ $a=16$ $c=-2$

$r = \frac{4}{16} = \frac{1}{4}$

$f(x) = 16\left(e^{-1.4}\right)^{\frac{x+2}{7}} - 10$

$(6, 28)$, $(11, 19)$



$d=30$

$c=6$ $a=-2$

$r = \frac{11}{2}$

$g(x) = -2\left(\frac{11}{2}\right)^{\frac{x-6}{5}} + 30$

$= -2\left(e^{1.7}\right)^{\frac{x-6}{5}} + 30$

Practice Problems: 7.2 page 354-357 # 1-8, 10