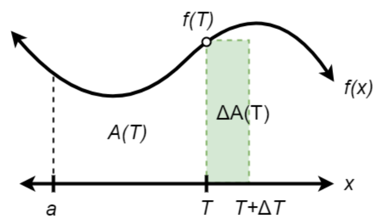
5.4 - Antiderivative and Fundamental Theorem of Calculus

Mr. Guillen's AP Calculus

Our goal is to define the integral of f as the *antiderivative* of f. To do this we will have to consider how the area changes as the interval changes. Begin by letting $A(T)$ be the area under $f(x)$ on the interval $[a, T]$, where a is fixed and T is free to change.

$$A(T) = \int_a^T f(x)dx$$

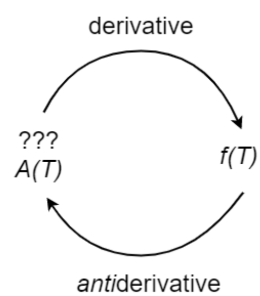
Therefore, when T changes by a small amount, say ΔT, then the area changes by small amount that can be approximated by a rectangle: $\Delta A(T) = f(T) \cdot \Delta T$.



Notice that we can rearrange the change in area to get

$$\frac{\Delta A(T)}{\Delta T} = f(T)$$

By letting $\Delta T \to 0$, we get that the derivative of A with respect to T is the function f, $\dfrac{dA}{dT} = f(T)$. Let's stop there and think about this statement. This means that that area (or the integral) under f is some function A such that if we take the derivative of A we get the function f. To solve this we have to work backwards to find A, and therefore say that A is an "anti"derivative of f.



Notice that A is *an* antiderivative and not *the* antiderivative of f. To illustrate this, consider the fact that the derivative of x^2 and $x^2 + 1$ are both $2x$, since the derivative of a constant is 0. Therefore, the area under f (or the integral of f), will be any antiderivative of f plus some $c \in \mathbb{R}$.

$$A(T) = \int_a^T f(x)dx = F(T) + c \tag{1}$$

Where F is any antiderivative of f. For the remainder of this chapter I will use big F to indicate an antiderivative of little f, and use it to replace A when I refer to the area under f.

Example 1

a. $\displaystyle\int_0^T x dx = \frac{1}{2}T^2 + c$

b $\displaystyle\int_x^\pi \sin\theta d\theta = -\int_\pi^x \sin\theta d\theta = -(-\cos x + c') = \cos x + c$

c. $\displaystyle\int_{-5}^{\ln y} e^t dt = e^{\ln y} + c = y + c$

Fundamental Theorem of Calculus - Part 1

If f is a continuous function on $[a,b]$, let $F(x) = \displaystyle\int_a^x f(t)dt$. Then, assuming $\dfrac{dF}{dx}$ is well defined

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{2}$$

Proof. The textbook uses first principles and Mean Value Theorem to prove this result; however, the way we built the antiderivative makes this result almost obvious. We had defined $A(T) = \int_a^T f(x)dx$ and loosely derived the fact that $\dfrac{dA}{dT} = f(T)$. Therefore,

$$\frac{dA}{dT} = \frac{d}{dT}\int_a^T f(x)dx = f(T)$$

and we just need to do some relabelling of variables. □

Equation (2) formalizes the idea that the integral is the inverse opperation of the derivative, much like how operations such as multiplication and division are inverses.

Example 2

a. $\displaystyle\frac{d}{dx}\int_0^x t(8-t)dt = x(8-x)$

b. $\displaystyle\frac{d}{dx}\int_x^{2\pi} \sin^2\frac{r}{4}dr = -\frac{d}{dx}\int_{2\pi}^x \sin^2\frac{r}{4}dr = \sin^2\frac{x}{4}$ using integral properties.

c. Find $\dfrac{dy}{dx}$, if $y = \displaystyle\int_3^{x^2+x} \frac{1}{z+1}dz$,

By chain rule we know that $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$, so if we let $u = x^2 + x$ we get that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{du}\int_3^u \frac{1}{z+1}dz\cdot\frac{du}{dx}\\ &= \frac{1}{u+1}\cdot(2x+1)\\ &= \frac{2x+1}{x^2+x+1}\end{aligned}$$

Fundamental Theorem of Calculus - Part 2

If f is continuous on $[a, b]$ and if F is an antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b \tag{3}$$

Proof. We know from equation (1) that $F(t) + c = \int_a^t f(x)dx$, for any $t \in [a, b]$ and some fixed $c \in \mathbb{R}$. Therefore,

$$\int_a^a f(x)dx = F(a) + c = 0$$

Hence, $c = -F(a)$. All that remains is to let $t = b$.

$$\begin{aligned}\int_a^b f(x)dx &= F(b) + c \\ &= F(b) - F(a)\end{aligned}$$

□

Notice that equation (3) is just a special case of equation (1), as nothing new needed to be introduced. In fact, all of the Fundamental Theorem of Calculus came from the central idea of observing the rate of change of the area as the interval changed.

Example 3

a. $\int_3^6 (8x - x^2)dx = (4x^2 - \frac{1}{3}x^3)\Big|_3^6 = (144 - 72) - (36 - 9) = 45$

b. $\int_1^4 \frac{s^2 + \sqrt{s}}{s^2} ds = \int_1^4 (1 + s^{-3/2})ds = (s - 2s^{-1/2})\Big|_1^4 = (4 - 1) - (1 - 2) = 4$

c. $\int_{-4}^4 |x|dx = \int_{-4}^0 -xdx + \int_0^4 xdx = -\frac{x^2}{2}\Big|_{-4}^0 + \frac{x^2}{2}\Big|_0^4 = (0 - (-8)) + (8 - 0) = 16$