Extra Topic - Game Theory

Mr. Guillen's AP Calculus

Game theory is the method of studying situtations where two or more "players" make their decisions based on the decisions of others. This applies not just in sports and games (obviously) but in areas such as economics (two or more firms are in competition for business), politics (candidates are competing for votes), law (a jury deciding a verdict), and even biology (animal behaviour or evolution).

Simplifying this, game theory is the study of the interactions between organisms or *players* and how the actions or *strategies* of one player affect the other players.

Definition: A game in **strategic** or **normal form** has the following characteristics:

(i) There is a set of players $N = \{1, 2, \ldots, n\}$ where $n \geq 2$.

(ii) Player i has a set of strategies, S_i, available to them. We call $s_i \in S_i$ a strategy for player i.

(iii) Taking each player together, we call the set of **strategy profiles** $S = S_1 \times S_2 \times \cdots \times S_n$. An element of S would be of the form $s = (s_1, s_2, \ldots, s_n)$ which consists of one strategy per player.

(iv) The **payoff** of a strategy profile for player i is how they value the outcome of the game if that strategy profile is used. In other words, the payoff is how happy it makes player i. Formally, the payoff is a function from the set of strategy profiles to the set of real numbers

$$\pi_i : S \longrightarrow \mathbb{R}$$

We typically represent strategies and payoffs of a game using a payoff matrix, where the strategies of the first player are put along the rows and the strategies of the other players on along the columns. Inside the payoff matrix, the tuple (x, y) is the payoff for player X and Y respectively.

		Player Y	
		A	B
Player X	A	(x, y)	(x, y)
	B	(x, y)	(x, y)

Dominant Strategy

Consider the following game called "Golden Balls" that has a prize of $1,000. Two players are given two golden balls each with the words 'steal', T, and 'share', H, written on the inside. Each player picks a ball in secret and then they reveal their choices simultaneously. If both players pick H, then they both get $500. If one person picks T and the other H, then the player that picked 'steal' will get all $1,000 and the other player gets nothing. And if they both pick T, then they each get nothing.

Example 1 - The Average Player

To analyze the game, we need to asign payoffs to each strategy profile. In this first example we will have that each player has the preference that they want as much money as possible, but want to avoid and the feeling of betrayal that could happen if they choose to share the money and their opponent stole it. From the perspective of player 1, they could give the possible payoffs to the different strategy profiles.

$\pi_1(T, H) = 2$ win \$1,000 $\qquad$ $\pi_1(H, H) = 1$ win \$500

$\pi_1(T, T) = 0$ win \$0 $\qquad$ $\pi_1(H, T) = -1$ win \$0 and get betrayed

Notice that since $2 > 1 > 0 > -1$ there is a well defined order of the payoffs, and it is very clear what outcomes the player prefers.

As a payoff matrix, we can illustrate this as

		Player 2	
		T	H
Player 1	T	$(0,0)$	$(2,-1)$
	H	$(-1,2)$	$(1,1)$

To analyze what the players should do, notice that if Player 2 picks T, then Player 1 will have a higher payoff if they pick T too (-1 vs 0). Likewise, if Player picks H, then Player 1 should pick T to have a higher payoff (2 vs 1). In both cases, Player 1 should pick T to maximize their payoff. This is called a **strictly dominant strategy** and is characterized as the best strategy regardless of what the other players do.

Mathematically, we would say that strategy s_1^* is strictly dominant if $\pi_1(s_1^*, s_2) > \pi_1(s_1, s_2)$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

With the same argument, Player 2 should also pick T. Even though this will lead to the players picking the only option where no one wins money, this is the only rational option to both players as **no rational person will pick a strategy that is dominated** (note the 'ed').

Example 2 - The Apathetic Player

Suppose we change the payoffs for both players so that they are indifferent between losing together and losing alone.

$$\pi_1(T,H) = 2 \quad \text{win \$1,000} \qquad\qquad \pi_1(H,H) = 1 \quad \text{win \$500}$$
$$\pi_1(T,T) = 0 \quad \text{win \$0} \qquad\qquad \pi_1(H,T) = 0 \quad \text{win \$0}$$

		Player 2	
		T	H
Player 1	T	$(0,0)$	$(2,0)$
	H	$(0,2)$	$(1,1)$

In this example we see that if Player 2 picks T, then Player 1 does not have a preference to either strategy. However, if Player 2 picks H, then Player 1 should pick T to maximize their payoff. This is a **weakly dominant** strategy and is very similar to the above example.

We say that strategy s_1^* is weakly dominant if $\pi_1(s_1^*, s_2) \geq \pi_1(s_1, s_2)$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Example 3 - The Empathetic Player

This time, we will change the payoffs for each player to reflect a more considerate the average player. These players realize they would feel awful if they won \$1,000 by lying to the other player and would rather have won nothing.

$$\pi_1(T,H) = -1 \quad \text{win \$1,000 but feel guilty} \qquad\qquad \pi_1(H,H) = 1 \quad \text{win \$500}$$
$$\pi_1(T,T) = 0 \quad \text{win \$0} \qquad\qquad \pi_1(H,T) = -1 \quad \text{win \$0 and feel betrayed}$$

		Player 2	
		T	H
Player 1	T	$(0,0)$	$(-1,-1)$
	H	$(-1,-1)$	$(1,1)$

In this example, there is no dominant strategy. Both players want to match each other and there is no rational strategy unless we know what one player's tendency might be. Instead there are two mutually preffered strategy profiles: (H, H) and (T, T).

Nash Equilibrium

This idea of preffered strategy profiles between players leads us to one of the central ideas of Game Theory.

Definition: A **Nash equilibrium** is a strategy profile such that each player's strategy is a best response (has the highest payoff) against the strategies of the other players. In other words, if $(s_1^*, s_2^*) \in S_1 \times S_2$ is a Nash equilibrium, then:
(i) for every $s_1 \in S_1$ we have that $\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*)$.
(ii) for every $s_2 \in S_2$ we have that $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, s_2)$.

In Example 3 above, the strategy profiles of (T, T) and (H, H) are Nash equilibriums, because if one player changed strategies then both players would have a lower payoff.

Note the subtle difference between a dominant strategy and a Nash equilibrium. In a Nash equilibrium, the player's payoffs are maximized only by fixing the other players strategy, whereas a dominant strategy maxiizes payoffs regardless of the other players strategies.

With this idea in mind it is relatively simple to find Nash equilibria. Simply go through the rows and columns and find the strategy that is best for that player. Then the strategy profiles that are mutually optimal for both players will be a Nash equilibrium.

Example 4 Find the Nash equilibria for the following game:

		Player 2			
		a	b	c	d
Player 1	A	(4,0)	(3,2)	(2,2)	(4,1)
	B	$(4,2)$	$(2,1)$	(1,2)	(0,2)
	C	(3,5)	(5,5)	(3,1)	(5,0)
	D	(2,3)	(3,2)	(1,2)	(2,3)

When Player 2 picks a, Player 1 has the highest payoff when they pick A or B. In all other cases Player 1 is best to pick C. From the other perspective, we see that Player 2 should pick b or c if A is picked, not b if B is picked, a or b if C is picked, and a or d if D is picked. Taken together, the strategy profiles that are Nash equilibria are (B, a) and (C, b).

Common Knowledge

While we have found the strategy profiles that are in each player's best interest, it is natural to ask if there are any dominant strategies in the above game. We conclude this discussion by thinking about how completely rational players behave.

In the game in Example 4, we see that if Player 2 believes Player 1 to be a rational player (that is they will pick the strategy that is best for them), then Player 1 will never choose strategy D, since C will always give them a better payoff. We say that C strictly dominates D. Remember that a rational player will never pick a dominated strategy.

		Player 2			
		a	b	c	d
Player 1	A	(4,0)	(3,2)	(2,2)	(4,1)
	B	$(4,2)$	$(2,1)$	(1,2)	(0,2)
	C	(3,5)	(5,5)	(3,1)	(5,0)
	D	(2,3)	(3,2)	(1,2)	(2,3)

This reduces the game to

		Player 2			
		a	b	c	d
Player 1	A	(4,0)	(3,2)	(2,2)	(4,1)
	B	$(4,2)$	$(2,1)$	(1,2)	(0,2)
	C	(3,5)	(5,5)	(3,1)	(5,0)

Player 2 also sees that A weakly dominates B, so because Player 1 is rational they would never pick B.

		Player 2			
		a	b	c	d
Player 1	A	(4,0)	(3,2)	(2,2)	(4,1)
	C	(3,5)	(5,5)	(3,1)	(5,0)

If Player 1 knows that Player 2 believes them to be rational then Player 1 can assume that they have reduced the game to this equivalent game. They can see that b weakly dominates the other strategies. Knowing that Player 2 will pick b, Player 1 should pick C to maximize their payoff.

You could imagine that this thought process goes back and forth between the two players quite a bit for more complicated examples, each player thinking how much the other knows and adjusting their strategy accordingly. If this goes on forever, we say that the players have **common knowledge**. That is, each player knows exactly the same information.

References

Bonanno, G. 2018. *Game Theory.* California: CreateSpace Independent Publishing Platform.

Jackson, M. 2011. *A Brief Introduction to the Basics of Game Theory.* Stanford: Stanford University Press.

Polak, B. 2007. "Introduction: Five First Lessons." Lecture, Game Theory (ECON 159), Yale University, New Haven, CT, September 2007.

Practice Questions

1. What is the dominant strategy or strategies, if any, for each player in the following games of Golden Balls.
   (a) The Average Player and the Apathetic Player
   (b) The Average Player and the Empathetic Player
   (c) The Apathetic Player and the Empathetic Player
2. Prove that if a strategy profile is dominant then it is a Nash equilibrium.
3. There are two players. Each player is given an unmarked envelope and asked to put in it either nothing or \$300 of his own money or \$600 of his own money. A referee collects the envelopes, opens them, gathers all the money, then adds 50% of that amount (using his own money) and divides the total into two equal parts which he then distributes to the players.
   (a) Suppose that Player 1 has some animosity towards the referee and ranks the outcomes in terms of how much money the referee loses (the more, the better), while Player 2 is selfish and greedy and ranks the outcomes in terms of her own net gain. Represent the corresponding game using a table.
   (b) Is there a strictly dominant-strategy equilibrium?
   (c) Find the Nash equilibria.
4. Antonia and Bob cannot decide where to go to dinner. Antonia proposes the following procedure: she will write on a piece of paper either the number 2 or the number 4 or the number 6, while Bob will write on his piece of paper either the number 1 or 3 or 5. They will write their numbers secretly and independently. They then will show each other what they wrote and choose a restaurant according to the following rule: if the sum of the two numbers is 5 or less, they will go to a Mexican restaurant, if the sum is 7 they will go to an Italian restaurant and if the number is 9 or more they will go to a Japanese restaurant.
   (a) Suppose that Antonia prefers Mexican to Italian and Italian to Japanese, and Bob prefers Italian to Mexican to Japanese. Using payoff function with values 1,2, and 3 represent the game as a table.
   (b) For each player, is there any strictly or weakly dominanted strategies?
5. A mixed strategy is when there is no Nash equilibrium as in the case below. We can still determine an

		Player 2	
		a	b
Player 1	A	$(3,-3)$	$(0,0)$
	B	$(-1,1)$	$(2,-2)$

   equilibrium based on probability and expectation. Let p be the probability that Player 2 picks a and let q be the probability that Player 1 picks A. Then for an equilibrium to occur we need that

   $$\mathbb{E}[A] = \mathbb{E}[B] \quad \text{and} \quad \mathbb{E}[a] = \mathbb{E}[b]$$

   Determine the values of p and q.
6. Two players are blindfolded and have yellow party hats put on their head. Neither player can see their own hat but they can see each other's hat. Before they speak to each other, is it common knowledge that at least one person is wearing a yellow hat?
7. Imagine you are in a group of 100 people and playing the following game. Each player secretly choose an integer from 1 to 100 and submits it with their name. A player will win \$1,000 if they are the closest to the average number picked by everyone (if two or more people are the closest they will split the prize) and you win nothing otherwise. What would you pick and why?

8. Amaro is the captain of a pirate ship. His mateys, Bart, Charlotte, Daniel, and Eliza, are the other members of the crew. The group has come upon a bounty of 100 gold coins, and now must divide it up among the group according to "the pirate's code."

   The code stipulates that Amaro, as captain, gets to suggest the first plan for distributing the coins among the five pirates. After that proposal, each pirate (including Amaro) votes "yarr" or nay on whether to accept the proposal. If the proposal results in either a tied vote (equal numbers "yarr"/nay) or a majority of "yarrs," it passes and the coins are immediately distributed. If it fails to meet this threshold, Amaro must walk the plank, making Bart the next captain. (Amaro walking the plank removes him from future votes, as well as eligibility for coin disbursals, on account of his death. Yuck.)

   This process now repeats with Bart as captain, and the captain's hat will be passed on, in order, to Charlotte, Daniel, and finally Eliza. (If it gets all the way to Eliza without a passing proposal, she gets the booty.)

   To make the situation more complex, there are rules governing how the pirates act. First, they each want to stay alive (that's their highest priority), but their next priority is maximizing their personal gold horde. Second, they distrust each otherthere are no alliances and they cannot collaborate on a strategy. Third, they are bloodthirsty, and would love to see a fellow pirate walk the plank if they think it won't affect their own gold distribution. Fourth, each pirate has excellent logical deduction skills, and they're aware that everyone has the same skills. For the purposes of the puzzle, we can assume everyone is logical and obeys all the rules.

   What distribution should Amaro propose to ensure he lives and maximizes his own gold return?