Geometric Sequences and Series

KNOW	DO	UNDERSTAND
How to identify a sequence as geometric.	Build the equation for a geometric sequence and determine the sum of a geometric series	None yet

Vocab & Notation

- Common ratio

Definition: A **geometric sequence** is a sequence generated by multiplying the previous term by a fixed value.

next previous

$$a_{k+1} = a_k \cdot r \qquad , r \in \mathbb{R} \quad r \neq 0$$

$$\frac{a_{k+1}}{a_k} = \frac{a_k}{a_k} \cdot r$$

Definition: The **common ratio** is the ratio of consecutive terms in a geometric sequence.

$$\frac{a_{k+1}}{a_k} = r$$

true for all $k \in \mathbb{N}$

Let's look and see how a geometric sequence is built by letting $a_1 = A$

first term

$$(A, Ar, Ar^2, Ar^3, Ar^4, \dots) = (Ar^{k-1})_{k=1}^{\infty}$$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

$b_0 \quad b_1 \quad b_2 \quad b_3 \quad b_4$

common ratio

$$= (Ar^k)_{k=0}^{\infty}$$

Example: Determine sequences of the form $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ for the following patterns:

$$(2, 6, 18, \cdots)$$

×3 ×3

$a_1 = 2$ $\quad \frac{6}{2} = 3 = \frac{18}{6} = 3 = r$

$$(2 \cdot 3^{k-1})_{k=1}^{\infty} = (2 \cdot 3^k)_{k=0}^{\infty}$$

$$(100, -50, 25, \cdots)$$

$$\left(\frac{8}{9}, \frac{2}{9}, \frac{1}{18}, \cdots\right)$$

$a_1 = 8/9$ $\quad r = \frac{1}{4}$

$$\left(\frac{8}{9} \cdot \left(\frac{1}{4}\right)^{k-1}\right)_{k=1}^{\infty} = \left(\frac{8}{9 \cdot 4^{k-1}}\right)_{k=1}^{\infty}$$

$$= \left(\frac{8}{9 \cdot 4^k}\right)_{k=0}^{\infty}$$

$$\left(-9, -25, -\frac{625}{9}, \cdots\right)$$

$a_{k+1} = a_n \cdot r \Rightarrow ANS = ANS \cdot r$

** We can use our calculator to list the terms of a geometric sequence quickly using recursion.

Example: List the first 10 terms of the sequence where $a_1 = 10$ and $r = -\frac{4}{5}$

$(10, -8, 6.4, -5.12, 4.096, -3.2768, 2.62\ldots, -2.097, 1.67\ldots)$

$ANS = 10$ $\qquad$ $ANS \times (-4/5) = ANS$

Practice: List the first 10 terms of the sequence where $a_1 = 4$ and $r = \underline{1.06}$

☆ Newtons Method.

Now we would like to consider the geometric series, that is

$$\text{n terms} \leftarrow \left[S_n = \sum_{k=1}^{n} a_1 \cdot r^{k-1} \right.$$

start

common ratio

Oftentimes it is helpful to try and add or subtract copies of the summation to reduce it to something simpler.

$$S_n = a_1 + (a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1})$$

$$- \quad r S_n = (a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}) + a_1 r^n$$

$$(1-r) S_n = a_1 - a_1 r^n$$

$$\Rightarrow \boxed{S_n = \frac{a_1 (1 - r^n)}{1 - r}}$$

☆ Be careful about your start + end point

Example: Determine the following sums

$$S = \sum_{k=1}^{10} 4 \cdot 3^{k-1}$$

first term = 4

# of terms = 10 terms

$r = 3$ $> 1 \rightarrow$ BIG

$$S = \frac{4(1 - 3^{10})}{1-3} = 118094$$

$$S = \sum_{k=0}^{5} 3 \cdot \left(\frac{1}{2}\right)^k$$

first term = 3

# of terms = 6

$r = \frac{1}{2}$

< 1

$$S = \frac{3(1 - (1/2)^6)}{1 - 1/2} =$$

small

5.90625

Practice: Determine the following sums

$$\sum_{k=1}^{7} 8\cdot\left(-\frac{2}{3}\right)^{k-1}$$

$r = -2/3$

$$S = \frac{8\left(1 + \left(2/3\right)^7\right)}{1 + 2/3}$$

<1 → small

$$\sum_{k=0}^{6} \left(\frac{9}{4}\right)^{k}$$

$a_1 = 1$

$$S = \frac{1 - (9/4)^7}{1 - 9/4}$$

>1 → Big

Since we know the finite sum, we can consider what S_∞ would be:

$$S_n = \frac{a_1(1 - \cancel{r^n})}{1 - r}$$

→ small → so small its 0 when $n \to \infty$

$|r| < 1$

first term

$$\boxed{\frac{a_1}{1-r} = S_\infty} = \frac{a_1(1-0)}{1-r}$$

common ratio

Example: Determine the following sums

$$S = \sum_{k=1}^{\infty} 10\cdot\left(-\frac{2}{5}\right)^{k-1}$$

first $= 10$
$r = -2/5$

$$\Rightarrow S = \frac{10}{1 + 2/5} = \frac{50}{7}$$

$$\sum_{k=0}^{\infty} 7\cdot\left(-\frac{3}{4}\right)^{k}$$

first $= 7$ $r = -3/4$

$$S = \frac{7}{1 + 3/4} = 4$$

$$\sum_{k=2}^{\infty} 100\cdot\left(\frac{1}{2}\right)^{k}$$

first $= 25$ $r = 1/2$

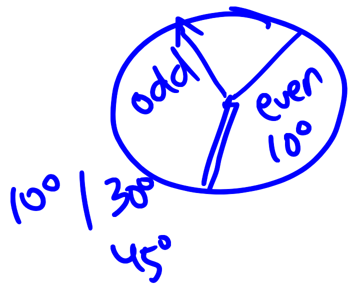
$$S = \frac{25}{1 - 1/2} = 50$$

$$\infty = \sum_{k=0}^{\infty} 12\cdot(1.01)^{k}$$

first 12
$r = 1.01$

$r > 1$
so does not work

Practice Problems: Handout sections III, V(c,d) Ryerson pdf: 1.3 # 1-7, 23 1.4 # 1-8, 16=19 1.5 # 1-7, 20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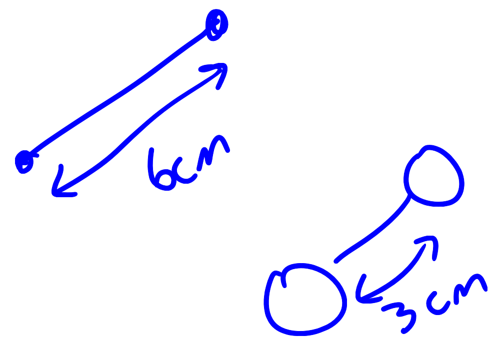
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