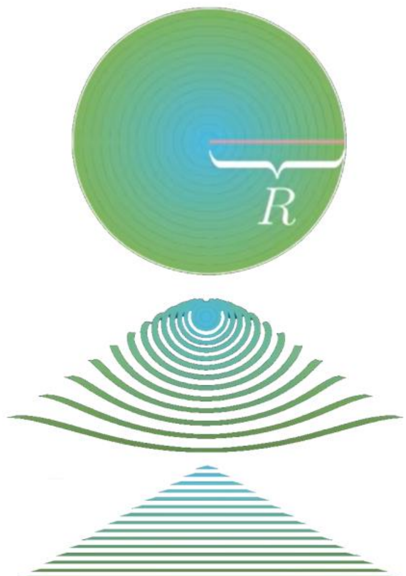
Area Under a Curve Part 1

Goal:
- Can define the area under any curve using a Riemann Sum and limit.

Terminology:
- Riemann Sum

Discussion question: Why is the area of a circle πr^2



We can cut the circle into thin circular strips. If the strips are thin enough, we can bend them into the shape of a line (like as thin as a wire).

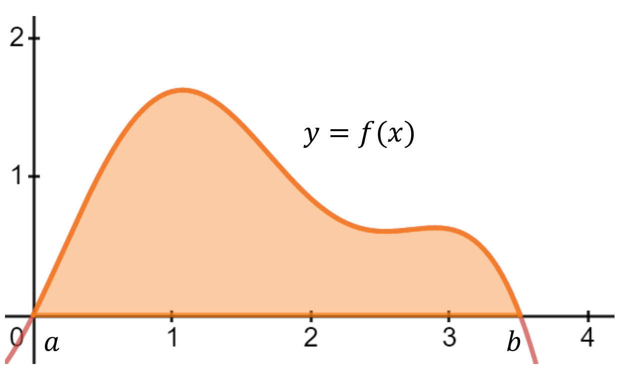
Once the strips are laid out on top of each other they will make a triangle with height R and the base will be the circumference $2\pi R$

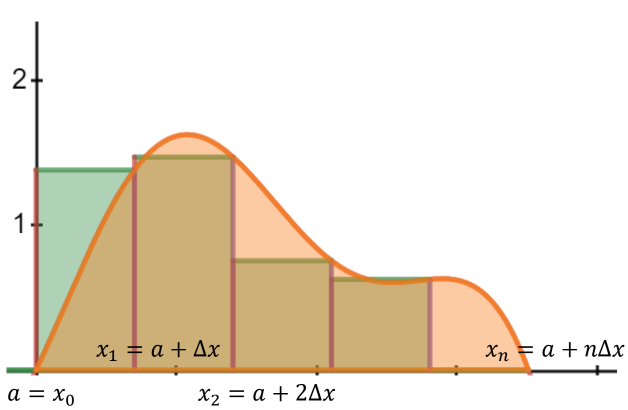
We can then determine the area of the circle as

$$A = \frac{R \cdot 2\pi R}{2} = \pi R^2$$

We are using rectangles to determine the area and if these rectangles are thing enough our method will give us the exact area.

Let's use this idea with an arbitrary shape.





If we were to approximate the area we might say the area is around 3 since the blob shape could be flattened to have a height of 1 and the base is pushed in to a width of 3. But to say more than that is difficult.

What we could do is cut the region into n rectangles so that each rectangle has width $\Delta x = \frac{b-a}{n}$. Then we have a choice of how tall we want the rectangles to be.

Pick some point $c_k \in [x_{k-1}, x_k]$ and then the height will be $f(c_k)$. Since it doesn't matter really where c_k is (as we want the rectangles to become very thin) we are going to pick the right end-point, x_k. Therefore the area of each rectangle is

$$f(x_k) \cdot \frac{b-a}{n}$$

And the sum of every rectangle is

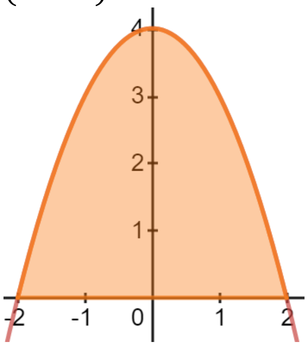
$$\sum_{k=1}^{n} f(x_k) \cdot \Delta x = \big(f(x_1) + f(x_2) + \cdots + f(x_n)\big)\Delta x$$

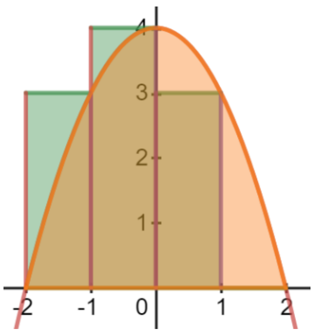
Note that $x_k = a + k \cdot \Delta x$

I have built a calculator for you to partition a region into n subintervals

https://www.desmos.com/calculator/t17czhwjyl

Example: Approximate the area under the parabola $f(x) = 4 - x^2$ on the interval $[-2, 2]$ using 4 subintervals $(n = 4)$.





If we make 4 subintervals, then each rectangle has width of $\frac{2-(-2)}{4} = 1$. And we partition $[-2, 2]$ into $\{-2, -1, 0, 1, 2\}$

The area is

$$\sum_{k=1}^{4} f(x_k) \cdot 1 = f(-1) + f(0) + f(1) + f(2)$$

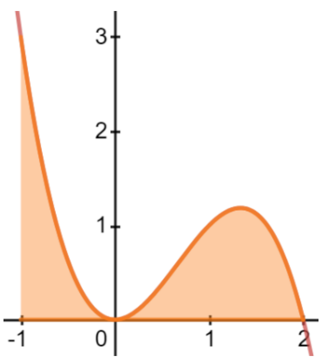
$$= 3 + 4 + 3 + 0 = 10$$

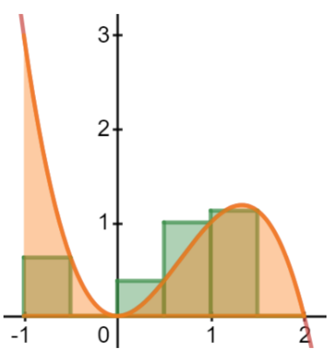
If we want the exact area, we need the limit as $n \to \infty$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(-2 + \frac{4k}{n}\right) \cdot \frac{4}{n} \approx 10.66(6) \text{ using } n = 100$$

Since $x_k = -2 + k \cdot \Delta x = -2 + k\left(\frac{4}{n}\right)$

Practice: Determine the area under the curve $f(x) = -x^3 + 2x^2$ on the interval $[-1, 2]$ using 6 subintervals





If we make 6 subintervals, then each rectangle has width of $\frac{2-(-1)}{6} = \frac{1}{2}$. And we partition $[-1, 2]$ into $\{-1, -0.5, 0, 0.5, 1, 1.5, \ 2\}$

The area is

$$\sum_{k=1}^{6} f(x_k) \cdot \frac{1}{2} = \big(f(-0.5) + f(0) + \cdots + f(2)\big) \cdot \frac{1}{2}$$

$$= (0.625 + 0 + 3.75 + 1 + 1.125 + 0) \cdot \frac{1}{2}$$

$$= 1.5625$$

If we want the exact area, we need the limit as $n \to \infty$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(-1 + \frac{6k}{n}\right) \cdot \frac{6}{n} \approx 2.2(1) \text{ using } n = 100$$

Since $x_k = -1 + k \cdot \Delta x = -1 + k\left(\frac{6}{n}\right)$

Note: In the regions that f is decreasing our rectangles are underestimating the area and vice versa in the regions f is increasing.

Practice Problems: 10.4 # 1, 3&4 (write the area as a limit $n \to \infty$ and approximate using $n = 4$, use the calculator to determine the area to 1 or 2 decimals of accuracy)

 # 6