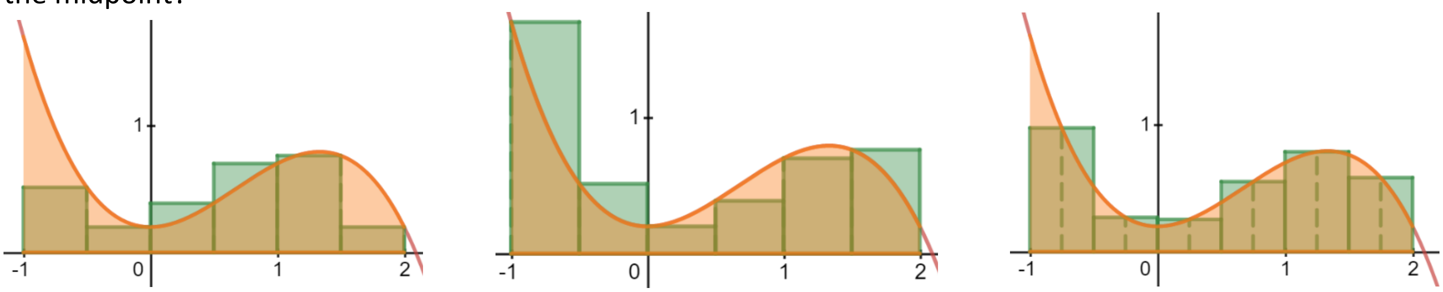
Area Under a Curve Part 2

Goal: • Can approximate the area under a curve using midpoint and trapezoids	
Terminology: • None	

Discussion question: Is using the average of the left and right enpoints to determine height the same as using the midpoint?



No. What the question is asking us is if

$$\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right)$$

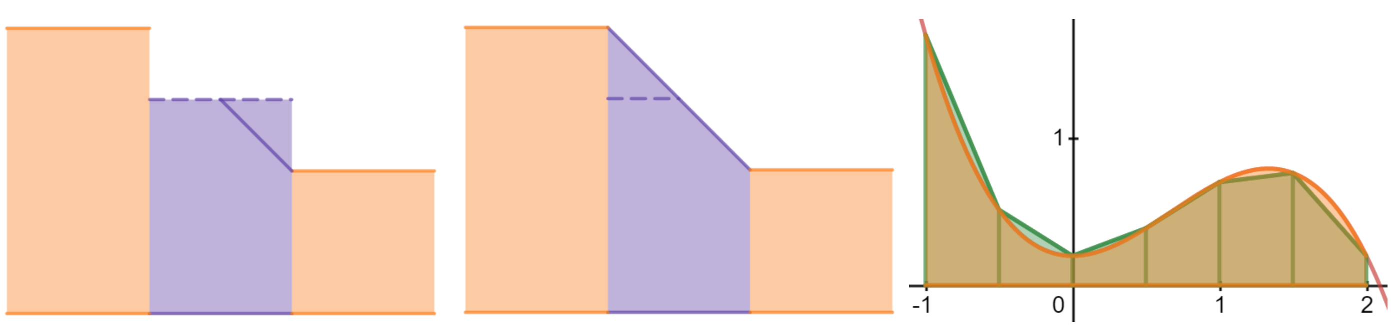
Is the average height at two endpoints equal to the height at the middle of the endpoints? This should feel like an unlikely thing to happen and all we need to do is pick really any function.

Let $f(x) = x^2$ and pick the interval $[0, 2]$. Is the average value of $f(0)$ and $f(2)$ equal to $f(1)$? Clearly no!

$$\frac{f(0)+f(2)}{2} = 2 \neq 1 = f(1)$$

So, what does the average value of the left and right give you? As illustrated below, the average height of two rectangles can be made into a trapezoid. So essentially the average of the left and right endpoint is the same as using triangles/trapezoids to approximate the area rather than rectangles.

It should be clear that using the midpoint or trapezoids gives a much better cover of the desired area.

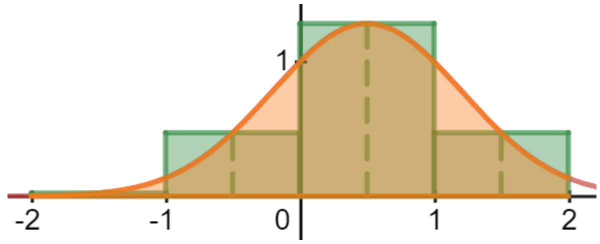


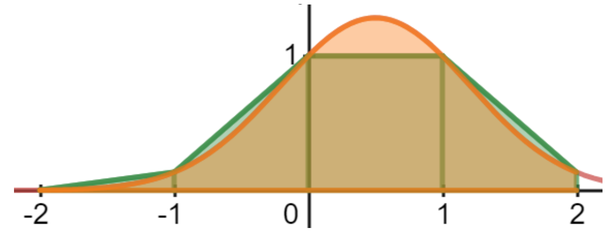
We have two new calculators to find the area

https://www.desmos.com/calculator/bilc70rubs

https://www.desmos.com/calculator/mkfcz7p8el

Example: Approximate the area under the curve $f(x) = e^{x-x^2}$ on the interval $[-2, 2]$ using 4 subintervals using rectangles with left, right, and middle endpoints AND seperately with trapezoids. Use the calculator to approximate the area with 100 subintervals. (The exact area is 2.23684246999…)





If we make 4 subintervals, then each rectangle has width of $\frac{2-(-2)}{4} = 1$. And we partition $[-2, 2]$ into

$$\{-2 = x_0, x_1, x_2, x_3, x_4 = 2\}, \quad \text{Note: } x_k = -2 + k$$

Using **Right Endpoint**

$$\sum_{k=1}^{4} f(x_k) \cdot \Delta x = f(-1) + f(0) + f(1) + f(2)$$
$$= e^{-2} + 1 + 1 + e^{-2} = 2.2706 \ldots$$

Using **Left Endpoint**

$$\sum_{k=0}^{3} f(x_k) \cdot \Delta x = f(-2) + f(-1) + f(0) + f(1)$$
$$= e^{-6} + e^{-2} + 1 + 1 = 2.1378 \ldots$$

Using **Midpoint**

$$\sum_{k=0}^{3} f(m_k) \cdot \Delta x = f(-1.5) + f(-0.5) + f(0.5) + f(1.5)$$
$$= e^{-3.75} + e^{-0.75} + e^{0.25} + e^{-0.75} = 2.2522 \ldots$$

Here: $m_k = x_k + 0.5\Delta x$ because we are using the left endpoint and shifting it half a width to the right.

Using **Trapezoids**

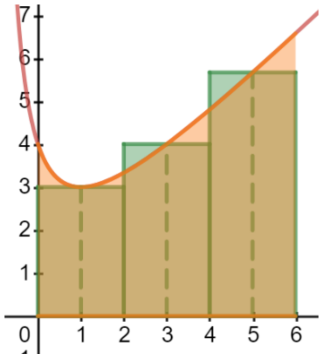
$$\frac{1}{2}\left(\sum_{k=1}^{4} f(x_k) \cdot \Delta x + \sum_{k=0}^{3} f(x_k) \cdot \Delta x\right) = \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{k=1}^{3} f(x_k) + f(x_4)\right)$$
$$= \frac{1}{2}\big(f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)\big)$$
$$= 2 + \frac{3}{2}e^{-2} + \frac{1}{2}e^{-6} = 2.2042 \ldots$$

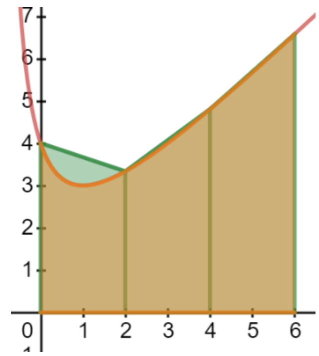
If we let $n = 100$ using the calculator we get

$$\text{Area}_{\text{mid}} = 2.2368703587 \ldots$$
$$\text{Area}_{\text{trap}} = 2.2367866885 \ldots$$
$$\text{Area}_{\text{true}} = 2.2368424999 \ldots$$

Notice that the trapezoids are underestimating the area because the curve is mostly concave down so the trapezoid is completely below the curve. Using the midpoint allows us to compensate over and under-estimating better as

Practice: Determine the area under the curve $f(x) = \frac{4}{x+1} + x$ on the interval $[0, 6]$ using 3 subintervals





If we make 3 subintervals, then each rectangle has width of $\frac{6}{3} = 2$. And we partition $[0, 6]$ into

$$\{0 = x_0, x_1, x_2, x_3 = 6\}, \qquad \text{Note: } x_k = 2k$$

Using **Right Endpoint**

$$\sum_{k=1}^{3} f(x_k) \cdot 2 = \big(f(2) + f(4) + f(6)\big) \cdot 2$$

$$= 2\left(\frac{4}{3} + 2 + \frac{4}{5} + 4 + \frac{4}{7} + 6\right) = 29.4095 \ldots$$

Using **Left Endpoint**

$$\sum_{k=0}^{2} f(x_k) \cdot 2 = \big(f(0) + f(2) + f(4)\big) \cdot 2$$

$$= 2\left(\frac{4}{1} + 0 + \frac{4}{3} + 2 + \frac{4}{5} + 4\right) = 24.2666 \ldots$$

Using **Midpoint**

$$\sum_{k=0}^{2} f(m_k) \cdot 2 = \big(f(1) + f(3) + f(5)\big) \cdot 2$$

$$= 2\left(\frac{4}{2} + 1 + \frac{4}{4} + 3 + \frac{4}{6} + 5\right) = 25.3333 \ldots$$

Here: $m_k = x_k + 1$

Using **Trapezoids**

$$\frac{\Delta x}{2}\left(f(x_0) + \sum_{k=1}^{2} f(x_k) + f(x_3)\right) = \big(f(0) + 2f(2) + 2f(4) + f(6)\big)$$

$$= \frac{4}{1} + 0 + 2\left(\frac{4}{3} + 2 + \frac{4}{5} + 4\right) + \frac{4}{7} + 6 = 26.8380 \ldots$$

If we let $n = 100$ using the calculator we get

$$\text{Area}_{\text{mid}} = 25.783053218 \ldots$$
$$\text{Area}_{\text{trap}} = 25.784815675 \ldots$$
$$\text{Area}_{\text{true}} = 25.783640596 \ldots$$

Notice that now the trapezoids are overestimating the area because the curve is mostly concave down so the trapezoid is completely below the curve. **In general midpoint will usually outperform trapezoid**.

Practice Problems: 10.5 # 1, 3, 6
Extra Practice Problems