Fundamental Theorem of Calculus: Part 1

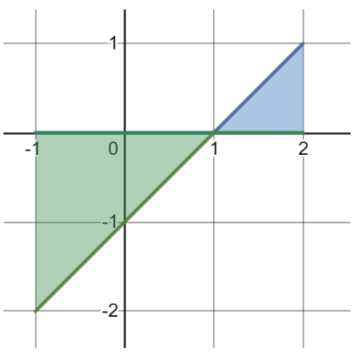
Goal:
• Understands why the integral is the antiderivative • Understands why the derivative of an integral is the integrand • Understands that the integral is the area under a curve
Terminology: • Fundamental Theorem of Calculus

Discussion: What is the value of the following limit?

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\cdot\Delta x$$

Where $f(x) = x - 1$ and $x_k = -1 + \frac{3k}{n}$ and $\Delta x = \frac{3}{n}$

Note that this is the area under the line $y = x - 1$ on the interval $[-1, 2]$ since $x_k = -1 + k\cdot\frac{3}{n}$ we see we start at -1 and move an interval length of 3 units. The question then is what is that area? The only thing to note is that part of the area is negative as it has negative height. Adding the positive and negative we get: $-2 + \frac{1}{2} = -1.5$



We are going to define new notation now for this limit. We saw at the start of the year that $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ so being consistent we have

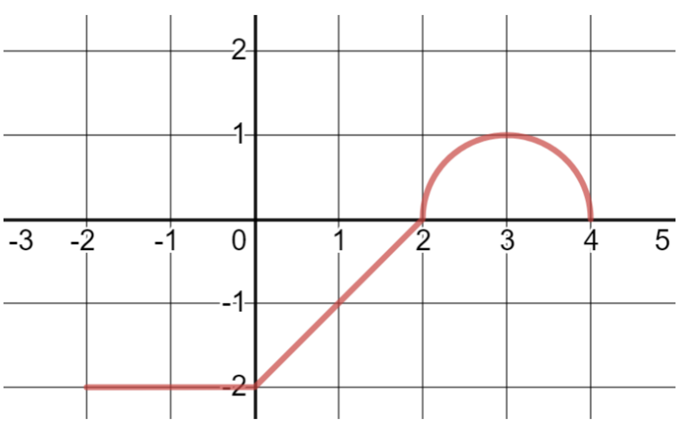
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\cdot\Delta x = \int_a^b f(x)dx$$

The x_k becomes continuous x and the capital Σ becomes a smoother $\int$. The a, b is there to tell you the interval you want the area of.

Example: Consider the function f below, and define

$$F(t) = \int_{-2}^{t} f(x)dx$$

Determine the following values: $F(-2),\ F(0),\ F(2),\ F(4)$



$$F(-2) = \int_{-2}^{-2} f(x)dx = 0, \qquad \text{(no width)}$$

$$F(0) = \int_{-2}^{0} f(x)dx = -4, \qquad \text{(rectangle)}$$

$$F(2) = \int_{-2}^{2} f(x)dx = -6, \qquad \text{(rectangle + triangle)}$$

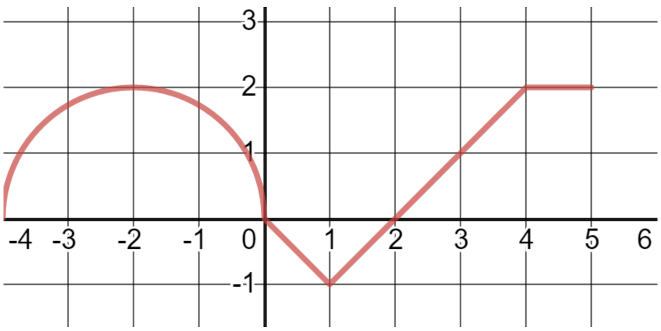
$$F(4) = \int_{-2}^{4} f(x)dx = -6 + \frac{1}{2}\pi$$

$$(\text{rectangle} + \text{triangle} + \frac{1}{2}\text{circle})$$

Practice: Given the function f below and the function

$$F(t) = \int_{-4}^{t} f(x)dx$$

Determine the following values: $F(-4), F(0), F(2), F(5)$



$$F(-4) = \int_{-4}^{-4} f(x)dx = 0, \qquad \text{(no width)}$$

$$F(0) = \int_{-4}^{0} f(x)dx = 2\pi, \qquad \left(\frac{1}{2}\text{circle}\right)$$

$$F(2) = \int_{-4}^{2} f(x)dx = 2\pi - 1, \quad (\frac{1}{2}\text{circle} + \text{ triangle})$$

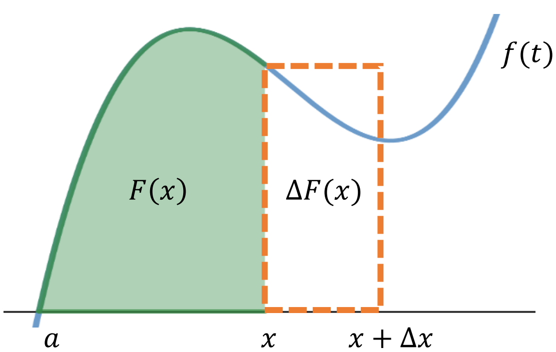
$$F(5) = \int_{-4}^{5} f(x)dx = 2\pi + 3,$$

$$(\frac{1}{2}\text{circle} + \text{triangle} + \text{trapezoid})$$

In general, if we have some function, f, and define a new function

$$F(x) = \int_{a}^{x} f(t)dt$$

We can consider what happens when we have a small change in x, and then consider what happens when $\Delta x \to 0$.



Note that we swapped the variables x and t since we are going to focus on the function defined as an integral

$F(x)$ is the shaded area and $f(t)$ is the curve bounding the area.

Make a small step of size Δx and consider the change in area $\Delta F(x)$

Here we can define $\Delta F(x)$ in two ways

$$\Delta F(x) = f(x) \cdot \Delta x$$

And

$$\Delta F(x) \approx F(x + \Delta x) - F(x), \qquad \text{(which gets better as } \Delta x \to 0\text{)}$$

$$\Rightarrow F(x + \Delta x) - F(x) \approx f(x) \cdot \Delta x$$

$$\Rightarrow \frac{F(x + \Delta x) - F(x)}{\Delta x} \approx f(x)$$

Let's take the limit and see that

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

$$\Rightarrow \frac{d}{dx} F(x) = f(x), \qquad \text{OR} \qquad \frac{d}{dx} \int_{a}^{x} f(t)dt = f(x)$$

This leads us to the first part of **Fundamental Theorem of Calculus:**

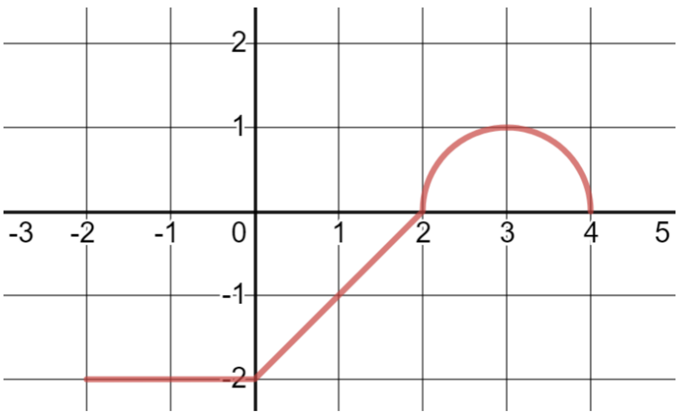
$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

This allows us to define the area under the curve or integral as the antiderivative of the function. It may not seem like much, but without this we would not have a physical meaning to the antiderivative.

Example: Given the function f below, and

$$F(x) = \int_{-2}^{x} f(t)dt$$

Determine where F has an extrema and inflection points.



Note that

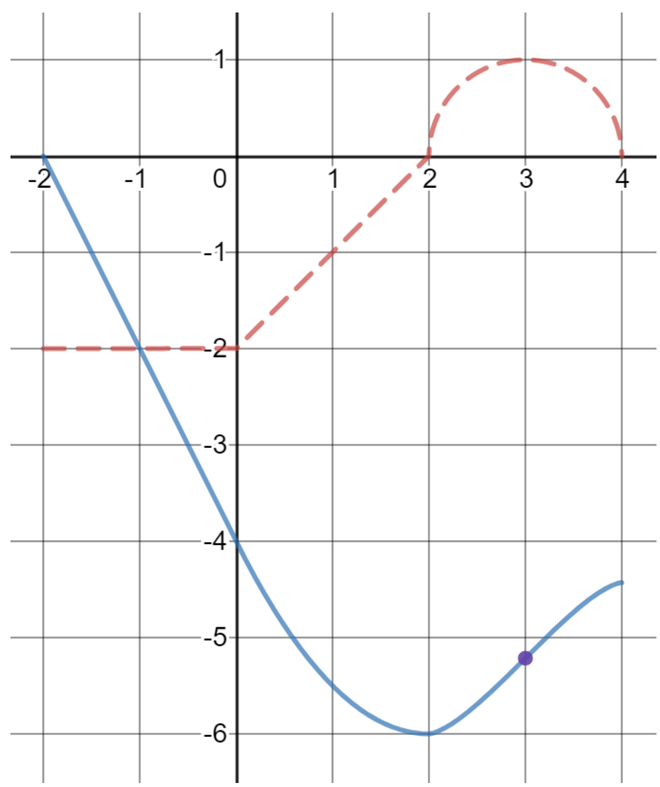
$$\frac{d}{dx}F(x) = f(x)$$

So, the graph we are looking at is just $F'(x)$ which we have spent time looking at before.

To find extremum, we need to look for where $F'(x)$ changes sign which happens at $x = 2$ (going from negative to positive so this is a local minimum). We need to check the endpoints too for absolute extremum.

We know $F(-2) = 0$ and $F(4) = \frac{\pi}{2} - 6$. Therefore, the absolute max is at $x = -2$ and the absolute min is also a local min at $x = 2$.

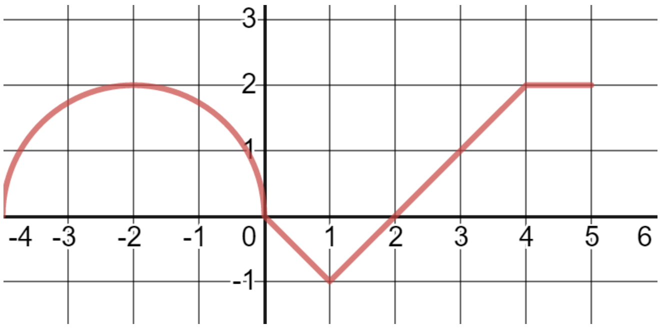
The inflection points are when $F''(x)$ changes sign, but since $F'(x) = f(x)$ we get $F''(x) = f'(x)$ so when does the slope of the curve we are looking at change sign? This will be at $x = 3$ when the slope goes positive (concave up) to negative (concave down). Note that on $[-2, 0]$ we gave that the concavity is 0 so the graph of F should be a straight line in that region.



Practice: Given the function f below and the function

$$F(x) = \int_{-4}^{x} f(t)dt$$

Determine where F has an extrema and inflection points



As before:

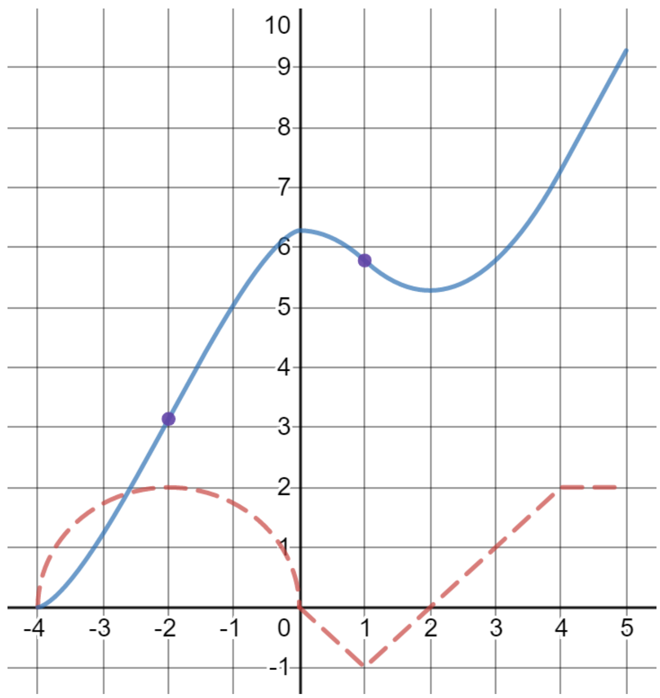
$$\frac{d}{dx}F(x) = f(x)$$

So, $F'(x) = f(x)$ and $F''(x) = f(x)$

To find extremum, we need to look for where $F'(x)$ changes sign which happens at $x = 0$ (going from positive to negative so this is a local max) and at $x = 2$ (other way so local min). We need to check the endpoints too for absolute extremum.

Using the geometry, we know that most of the area is positive and so $F(-4) = 0$ will be the absolute smallest (since $F(2) > 0$). As well, $F(5)$ will have the absolute largest area.

The inflection points are when $F''(x) = f'(x)$ changes sign, so when does the slope of the curve we are looking at change sign? This will be at $x = -2$ when the slope goes positive (concave up) to negative (concave down) AND at $x = 1$ from negative (concave down) to positive (concave up).



Practice Problems: Fundamental Theorem Practice Problems