Fundamental Theorem of Calculus: Part 2

Goal:
• Understands how to evaluate a definite integral for basic functions on $[a, b]$
Terminology:
• Total Area

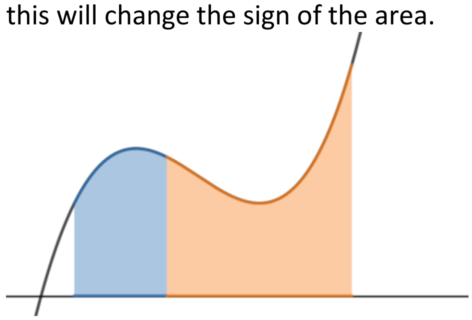
Discussion: Why are the following properties true?

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

For the first one, we are adding two areas that are side by side together. The area from $[a, c]$ is the same if we split it into two sections $[a, b]$ and $[b, c]$.

For the second one, we ask what does it mean to move backwards? Well then our $\Delta x < 0$ in the Riemann sum so this will change the sign of the area.



We want to evaluate the following integral where a, b could be anything on the domain of f.

$$\int_a^b f(x)dx$$

We are going to think of a, b as variables on both sides of some **fixed** point c. That is, let a, b be any numbers on the domain of f such that $a < c < b$

Then we can define a function (which is a very valuable trick to help make simplify a problem)

$$F(b) = \int_c^b f(x)dx$$

Where $F'(b) = f(b)$ so F is an antiderivative of f. Likewise,

$$F(a) = \int_c^a f(x)dx = -\int_a^c f(x)dx$$

$$\Rightarrow -F(a) = \int_a^c f(x)dx$$

We can stitch together these areas:

$$\int_a^c f(x)dx + \int_c^b f(x)dx = -F(a) + F(b)$$

$$\Rightarrow \int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example: Evaluate the following

$$\int_{-1}^{2}(x^2-4x+1)dx$$

We need to find an antiderivative to the function $f(x)=x^2-4x+1$. We see that such a function would be

$$F(x)=\frac{x^3}{3}-2x^2+x$$

$$\Rightarrow \int_{-1}^{2}(x^2-4x+1)dx=\frac{x^3}{3}-2x^2+x\Big|_{-1}^{2}$$

$$=\frac{8}{3}-8+2-\left(-\frac{1}{3}-2-1\right)$$

$$=3-6+3=0$$

Practice: Evaluate the following

$$\int_{0}^{1}\left(x^{99}+\frac{1}{\sqrt{3x+1}}\right)dx$$

We need to find an antiderivative to the function $f(x)=x^{99}+\frac{1}{\sqrt{3x+1}}$. We see that such a function would be

$$F(x)=\frac{x^{100}}{100}+\frac{2}{3}\sqrt{3x+1}$$

$$\Rightarrow \int_{0}^{1}\left(x^{99}+\frac{1}{\sqrt{3x+1}}\right)dx=\frac{x^{100}}{100}+\frac{2}{3}\sqrt{3x+1}\Big|_{0}^{1}$$

$$=\frac{1}{100}+\frac{4}{3}-\left(0+\frac{2}{3}\right)$$

$$=\frac{200}{300}+\frac{3}{300}=\frac{203}{300}$$

Practice: Evaluate the following

$$\int_{-4}^{-2}\frac{3-x}{x}dx$$

We need to find an antiderivative to the function $f(x)=\frac{3-x}{x}=\frac{3}{x}-1$. We see that such a function would be

$$F(x)=3\ln|x|-x$$

$$\Rightarrow \int_{-4}^{-2}\frac{3-x}{x}dx=3\ln|x|-x\Big|_{-4}^{-2}$$

$$=3\ln 2+2-(3\ln 4+4)$$

$$=3\ln 0.5-2$$

If we don't include the bounds we are shorthanding a way to write the general antiderivatvie of a function

$$\int f(x)dx = \int_0^x f(t)dt = F(x) + C$$

Example: Evaluate

$$\int \frac{x+1}{x-1} dx$$

We need to find an antiderivative to the function $f(x) = \frac{x+1}{x-1}$. It is NOT $\left(\frac{x^2}{2} + x\right) \ln|x-1|$ since that would require a product rule. Instead, we need to simplify using long division.

$$\frac{x+1}{x-1} = 1 + \frac{2}{x-1}$$

We can then see that an antiderivative would be

$$F(x) = x + 2\ln|x-1|$$

Check that $F'(x) = 1 + \frac{2}{x-1} = \frac{x-1+2}{x-1} = \frac{x+1}{x-1}$

$$\Rightarrow \int \frac{x+1}{x-1} dx = x + 2\ln|x-1| + C$$

Practice: Evaluate

$$\int \frac{2x^2 - x + 3}{x+1} dx$$

Using long division, we get that

$$\frac{2x^2 - x + 3}{x+1} = 2x - 3 + \frac{6}{x+1}$$

We can then see that an antiderivative would be

$$F(x) = x^2 - 3x + 6\ln|x+1|$$

Check that $F'(x) = 2x - 3 + \frac{6}{x+1} = \frac{2x^2+2x-3x-3+6}{x+1} = \frac{2x^2-x+3}{x+1}$

$$\Rightarrow \int \frac{2x^2 - x + 3}{x+1} dx = x^2 - 3x + 6\ln|x+1| + C$$

Practice Problems: 10.1 # 1 (NOT e, h, m, n, p)
11.2 # 1a-m, 2a-e, 3ab, 4
11.2 # Problem Plus