u-Substitution

Goal:
- Can identify a function and its derivative in an integral
- Understands substitution as reverse chain rule

Terminology:
- Substitution

Discussion: Given a function $y = f(x)$, how would you identify that f is the derivative of another function after applying chain rule?

Our goal today is to find the antiderivative of more complex functions.

If we see that our integral is of the form

$$\int f(g(x))g'(x)dx$$

What we are looking at is a function after chain rule: $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$ and F is an antiderivative of f.

$$\Rightarrow \int f(g(x))g'(x)dx = F(g(x)) + C$$

To make this integral easier to evaluate we let $u = g(x)$ so that

$$F(u) + C = \int f(u)du = \int f(u)g'(x)dx$$

and it looks like $du = g'(x)dx$.

Now we can imagine this is because $\frac{du}{dx} = g'(x)$ (since $u = g(x)$) and you multiply both sides by dx but that is not what happens! This is notation that comes from the Riemann sum where Δx was the width of the slices.
Since $\Delta x_k = x_{k+1} - x_k$ and $\Delta u_k = g(x_{k+1}) - g(x_k)$. We know

$$\lim_{n\to\infty} \frac{\Delta u_k}{\Delta x_k} = \lim_{x_{k+1}\to x_k} \frac{\Delta u_k}{\Delta x_k} = \lim_{x_{k+1}\to x_k} \frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k} = g'(x)$$

So

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(g(x_k))g'(x_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(g(x_k)) \frac{\overbrace{g(x_{k+1}) - g(x_k)}^{\Delta u_k}}{x_{k+1} - x_k} \Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(g(x_k))\, \Delta u_k$$

Example: Evaluate the following

$$\int (1+\ln x)e^{x\cdot\ln x}dx$$

We note that if $u = x\cdot\ln x$ then we have $\frac{du}{dx} = \ln x + 1$

$$\Rightarrow du = (\ln x + 1)dx \Leftrightarrow dx = \frac{du}{\ln x + 1}$$

So, we can rewrite the integral as

$$\int (1+\ln x)e^{x\cdot\ln x}dx = \int e^u du$$
$$= e^u + C$$
$$= e^{x\cdot\ln x} + C$$

Practice: Evaluate the following

$$\int \frac{2x-1}{x^2-x}dx$$

We should immediately see that the numerator is the derivative of the denominator

$$u = x^2 - x \Rightarrow \frac{du}{dx} = 2x - 1$$
$$du = (2x-1)dx \Leftrightarrow dx = \frac{du}{2x-1}$$

$$\int \frac{2x-1}{x^2-x}dx = \int \frac{1}{u}du$$
$$= \ln|u| + C$$
$$= \ln|x^2 - x| + C$$

Practice: Evaluate the following

$$\int \left(2+\frac{1}{\sqrt{x}}\right)\sqrt{x+\sqrt{x}}\,dx$$

We see that $2+\frac{1}{\sqrt{x}}$ looks a lot like the derivative of $x+\sqrt{x}$ just a difference of a multiple.

$$u = x + \sqrt{x} \Rightarrow \frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$
$$du = \left(1+\frac{1}{2\sqrt{x}}\right)dx \Leftrightarrow dx = \frac{2du}{2+\frac{1}{\sqrt{x}}}$$

$$\int \left(2+\frac{1}{\sqrt{x}}\right)\sqrt{x+\sqrt{x}}\,dx = \int \sqrt{u}\cdot 2du$$
$$= \frac{4}{3}u^{\frac{3}{2}} + C$$
$$= \frac{4}{3}(x+\sqrt{x})^{\frac{3}{2}} + C$$

If we are measuring the area under a curve and we use substituion, we need to keep in mind that the bounds of the interval depend on x and if we substitute we need to substitute everything.

Example: Evaluate the following

$$\int_{e^{-1}}^{e} \frac{(1+\ln x)^2(1-\ln x)}{x}dx$$

When looking for a good choice of u note that $1+\ln x$ is inside another function ($y = x^2$) and its derivative appears

$$u = 1+\ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

We need to get everything in terms of u which means the $1-\ln x$ and the bounds

$$\ln x = u-1, \qquad du = \frac{dx}{x} \Leftrightarrow dx = \frac{du}{x}$$

$$x = e \Rightarrow u = 2, \qquad x = e^{-1} \Rightarrow u = 0$$

So we replace the integral with

$$\begin{aligned}\int_{e^{-1}}^{e} \frac{(1+\ln x)^2(1-\ln x)}{x}dx &= \int_0^2 u^2(1-u+1)du \\ &= \int_0^2 (2u^2-u^3)du \\ &= \frac{2}{3}u^3 - \frac{1}{4}u^4\Big|_0^2 \\ &= \frac{16}{3} - 4 = \frac{4}{3}\end{aligned}$$

Practice: Evaluate

$$\int_0^1 x^3\sqrt{x^2+1}dx$$

Let $u = x^2+1$ since it is inside another function. Then $\frac{du}{dx} = 2x$ so $du = 2xdx \Leftrightarrow dx = \frac{du}{2x}$. Additionally, we have that $u(0) = 1$ and $u(1) = 2$

$$\int_0^1 x^3\sqrt{x^2+1}dx = \int_1^2 x^3\sqrt{u}\cdot\frac{1}{2x}du = \int_1^2 \frac{x^2\sqrt{u}}{2}\,du$$

Note that $x^2 = u-1$

$$\begin{aligned}\int_1^2 \frac{x^2\sqrt{u}}{2}\,du &= \frac{1}{2}\int_1^2 (u-1)\sqrt{u}\,du \\ &= \frac{1}{2}\int_1^2 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du \\ &= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2 \\ &= \left(\frac{4}{5}-\frac{2}{3}\right)\sqrt{2} - \left(\frac{1}{5}-\frac{1}{3}\right) \\ &= \frac{2}{15}(\sqrt{2}+1)\end{aligned}$$

Practice Problems: 11.3 # 1b, 2, 3 (NOT e, g, L, n-r), 4 (NOT d, e)

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