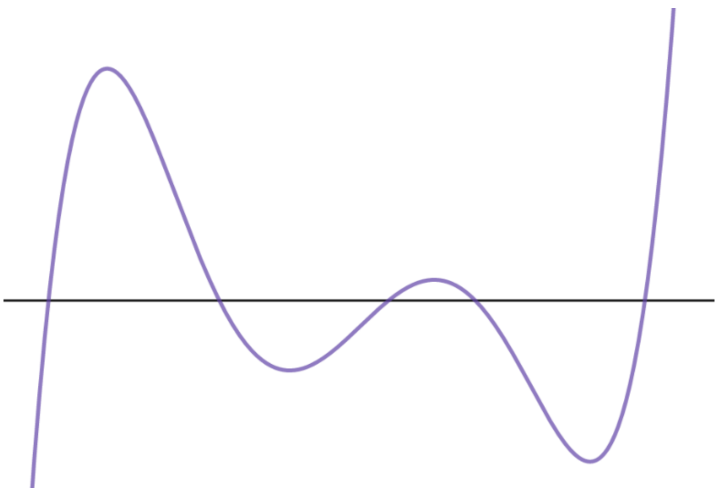
The Definite Integral

Goal: • Can write the limit of a Riemann Sum as a definite integral • Can evaluate definite integrals using signed area under the curve • Can use calculator to evaluate definite integrals
Terminology: • Integral
Reminder: • Quiz Wednesday January 15th

Riemann Sums:



Consider the function f and we want to find the net area under the curve on $[a, b]$. In general what we can do is consider some **partition** of $[a, b]$. That is, divide the interval intosome sequence

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

where $x_0 = a$ and $x_n = b$ and $x_k < x_{k+1}$.

With that our job is just to estimate the area under the curve on the subinterval $[x_k, x_{k+1}]$. If the partition is uniformaly small enough (that is the largest subinterval has small length) then any regtangle in the subintervals will be a good approximation to the area.

So pick some arbitrary point in the subinterval, $c_k \in [x_k, x_{k+1}]$ and use $f(c_k)$ to make the height of the rectangle which has width $\Delta x_k = x_{k+1} - x_k$, hence it has area of

The net area will be

This is a general Riemann Sum. Observe that RAM uses $\Delta x = \frac{b-a}{n}$ and $x_{k+1} = x_k + \Delta x$ for each k. The differences in LRAM, MRAM, and RRAM come from the choice of c_k.

For this sum to be the exact area under the curve we would need the largest length to go to zero, that is $\|P\| \to 0$

This is the definite integral. Just like the slope at a point was defined using limits, the area between curves also uses limits. This is calculus in its essence: How can we measure change at an instant? How can adding 0 infinitely often give us a real number?

Example: Evaluate the following

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n}\sqrt{16-c_k^2}\cdot \Delta x_k + \lim_{\|P\|\to 0}\sum_{k=1}^{n}\left(\frac{c_k}{2}+1\right)\cdot \Delta x_k$$

Where P is any arbitrary partition of $[-4, 4]$

Check on calculator! Using

$$\text{fnInt}(y_1, x, a, b)$$

OR on the graph you can plot the definite integral using "Calc" and entering the left and right bounds.

Example: Evaluate the following

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} c_k^2\cdot \sin\,(c_k)\cdot \Delta x_k$$

Where P is some partition of $[-1, 1]$

Practice Problems: 5.2 # 1-6 (and change to a closed form Riemann sum), 29-40

Pick and choose out of the following sets: 7-12, 13-22, 23-28

 Extra Riemann Sums

Look Ahead: 5.3 How can we define the average value of the function?

Extra Riemann Sums (Hard)

For each of the following, write it as a definite integral and then estimate using MRAM and Trapezoid method ($n = 4$) and the exact using fnInt. Recall that $\Delta x = (b - a)/n = \Delta x_k$ for all k in a regular partition.

1.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{3k}{n}\right)^2 + \left(\frac{3k}{n}\right) - 2 \right) \cdot \frac{3}{n}$$

2.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{2k}{n} + 1\right) \cdot \frac{2}{n}$$

3.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin^2\left(\frac{\pi k}{n}\right)}{n}$$

4.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{20e^{-\left(\frac{10k}{n}-5\right)^2}}{\sqrt{\pi}\cdot n}$$

Solutions to integrands and intervals (Note the solutions are NOT unique!)

1. $f(x) = x^2 + x - 2$ on $[0, 3]$
2. $f(x) = \ln x$ on $[1,3]$ OR $f(x) = \ln(x+1)$ on $[0,2]$
3. $f(x) = \sin^2(\pi x)$ on $[0,1]$
4. $f(x) = \frac{e^{-\left(\frac{x}{2}\right)^2}}{\sqrt{\pi}}$ on $[-10, 10]$ OR $f(x) = \frac{2e^{-x^2}}{\sqrt{\pi}}$ on $[-5, 5]$

In Class Evidence

Express the limit as an integral and use fnInt to evaluate it:

1. $$\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \cdot \Delta x_k$$

P is any partion of $[0, 2]$

4. $$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \cdot \Delta x_k$$

P is any partion of $[2, 3]$

Using graphs and the fact that

$$\int_0^1 x^3 dx = \frac{1}{4}$$

Evaluate the following integrals.

33. $$\int_0^1 (1 - x^3) dx$$

34. $$\int_0^1 (|x| - 1)^3\, dx$$

35 $$\int_0^2 \left(\frac{x}{2}\right)^3 dx$$