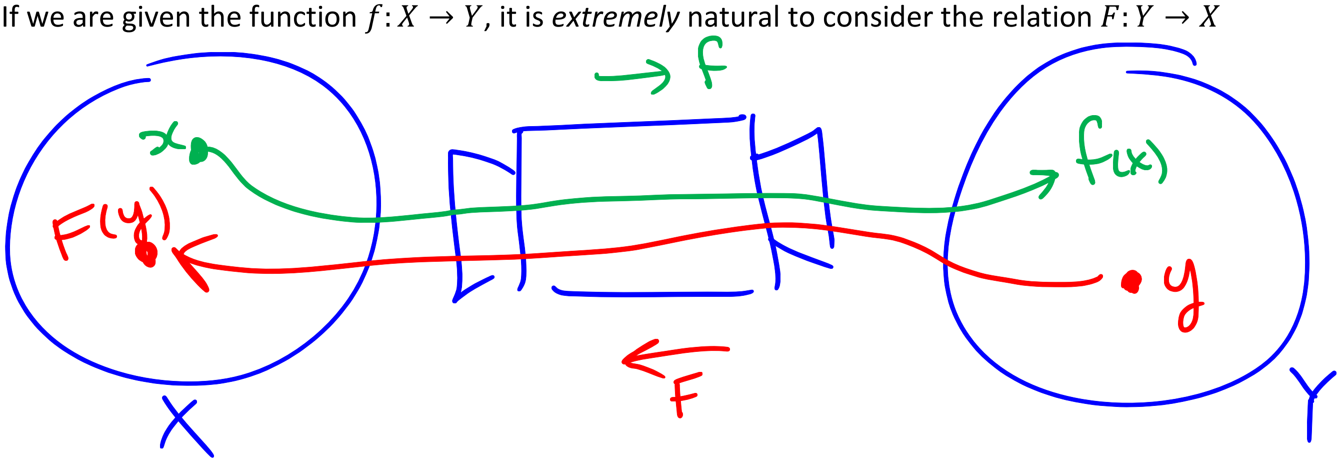
Inverses

KNOW	DO	UNDERSTAND
Be able identify when a function was inverted. Be able to recognize an inverse given the graphs	Use Desmos and Geogebra to graph inverses. Graph an inversion accurately by hand. Algebraically solve for an inverse. Algebraically confirm two functions are inverses.	*Inverses*: Can justify how the inverse will be transformed and what it's characteristics will be.

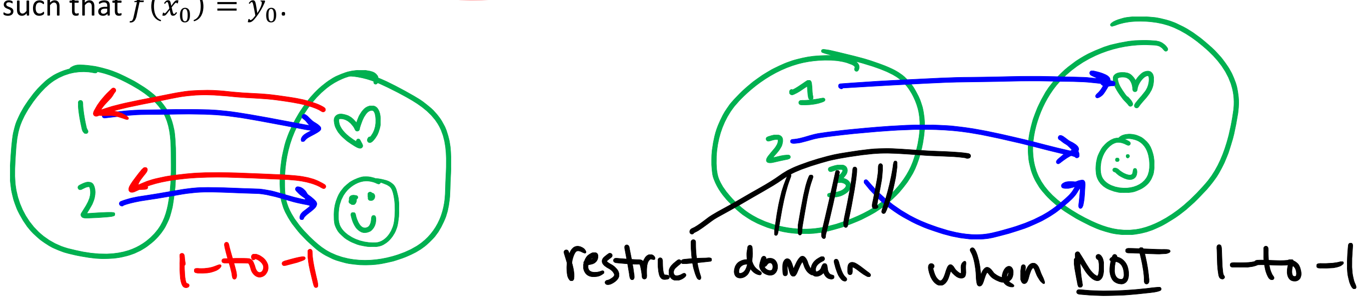
Vocab & Notation

- One-to-one: 1-to-1
- Inverse: f^{-1}

If we are given the function $f: X \to Y$, it is *extremely* natural to consider the relation $F: Y \to X$



Definition: A given function $f: x \mapsto y$ is **one-to-one** (1-to-1) if we have that for every $y_0 \in Y$ there is only one $x_0 \in X$ such that $f(x_0) = y_0$.



Definition: Given a 1-to-1 function f, we say that f^{-1} is an **inverse** of f if we have that

$$\cancel{f}(\cancel{f^{-1}}(x)) = x \leftarrow \text{identity function}$$

$g(x) = x$

IN out same

Example: Consider $f(x) = \frac{1}{x-3}$ and $f^{-1}(x) = \frac{1}{x} + 3$

$$f(f^{-1}(x)) = f\left(\frac{1}{x}+3\right) = \frac{\cancel{1}}{\cancel{\frac{1}{x}}\cancel{+3}\cancel{-3}} = x \checkmark$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x-3}\right) = \frac{\cancel{1}}{\frac{\cancel{1}}{x-\cancel{3}}}\cancel{+3} = x \checkmark$$

$f(x) = y$

Graphically, we can view the inverse in relation to the original graph by looking at the transformation from $f: x \mapsto y$ to $f^{-1}: y \mapsto x$. Or in other words from

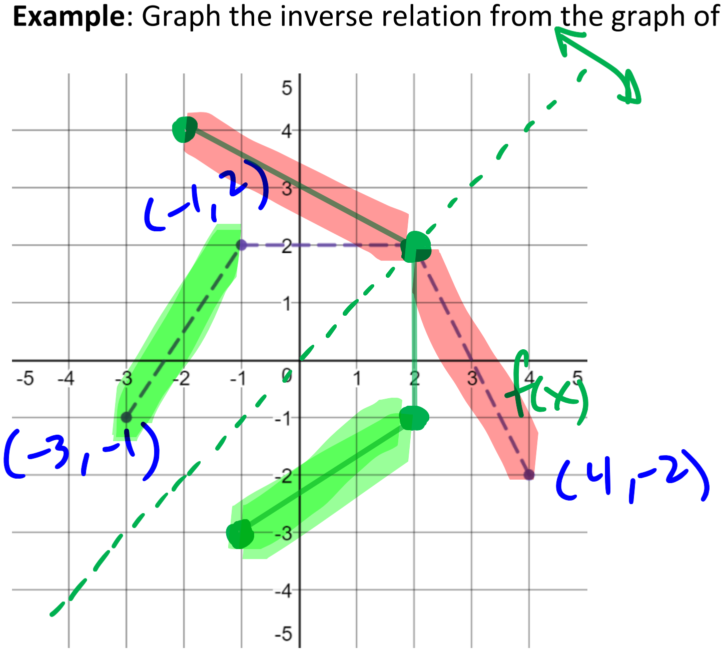
(x, y) (y, x)

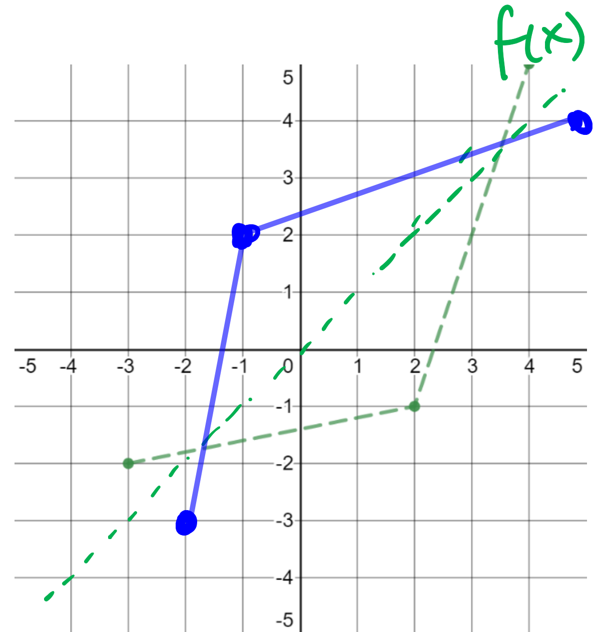
$I: (x, y) \mapsto (y, x)$

swap x and y

swap horizontal and vertical

Example: Graph the inverse relation from the graph of f below





$I: (-3, -1) \mapsto (-1, -3)$

NOT 1-to-1 fails horizontal line test

$f(-x) = f(x)$

* even functions not 1-to-1

1-to-1 passes horizontal line test

$\Rightarrow f^{-1}$ will pass vertical line test

So the final question is how do find the inverse function algebraically? Well, we want to look at what happens if y is the input and x is the output.

Example: For the above function $f(x) = 2x - 3$ we want to solve for x.

1.) x by 2
2.) − by 3
$\downarrow f$

$\uparrow f^{-1}$
1.) + by 3
2.) ÷ by 2

$f^{-1}(x) = \frac{x+3}{2}$

$y = 2x - 3$

$\Rightarrow f^{-1}(x) = \frac{x+3}{2}$

$y + 3 = 2x$

$\frac{y+3}{2} = x$ shape of f^{-1}

Example: If g is 1-to-1 then find the inverse of $f(x) = \frac{1}{4g(x+3)} + 2$

$$y = \frac{1}{4g(x+3)} + 2$$

$$y - 2 = \frac{1}{4g(x+3)}$$

$$\frac{1}{y-2} = 4g(x+3)$$

$$g^{-1}\left(\frac{1}{4(y-2)}\right) = \cancel{g^{-1}}\left(\cancel{g}(x+3)\right)$$

$$g^{-1}\left(\frac{1}{4(y-2)}\right) - 3 = x$$

$$\Rightarrow f^{-1}(x) = g^{-1}\left(\frac{1}{4(x-2)}\right) - 3$$

Practice: Find the equation of the inverse of the following functions (assume g is 1-to-1)

$$f(x) = \frac{x-1}{3}$$

$$y = \frac{x-1}{3}$$

$$3y + 1 = x$$

$$\Rightarrow f^{-1}(x) = 3x + 1$$

$$f(x) = \frac{1}{4}x^3 + 3$$

$$y = \frac{1}{4}x^3 + 3$$

$$\sqrt[3]{4(y-3)} = \sqrt[3]{x^3}$$

$$\sqrt[3]{4(x-3)} = f^{-1}(x)$$

$$f(x) = g\left(\frac{3}{2x-4}\right) + 1$$

$$y = g\left(\frac{3}{2x-4}\right) + 1$$

$$g^{-1}(y-1) = g^{-1}\left(g\left(\frac{3}{2x-4}\right)\right)$$

$$g^{-1}(y-1) = \frac{3}{2x-4}$$

$$\frac{1}{g^{-1}(y-1)} = \frac{2x-4}{3}$$

$$\Rightarrow \frac{1}{2}\left(\frac{3}{g^{-1}(y-1)} + 4\right) = x \Rightarrow f^{-1}(x) = \frac{3}{2g^{-1}(y-1)} + 2$$

$$f(x) = \frac{g(0.5x) - 1}{2}$$

$$y = \frac{1}{2}\left[g\left(\frac{x}{2}\right) - 1\right]$$

$$g^{-1}(2y+1) = g^{-1}\left(g\left(\frac{x}{2}\right)\right)$$

$$2g^{-1}(2y+1) = x$$

$$\Rightarrow f^{-1}(x) = 2g^{-1}(2x+1)$$

$$f(x) = g\left(3 + g^{-1}\left(\frac{2}{3x}\right)\right) - 2$$

$$y = g\left(3 + g^{-1}\left(\frac{2}{3x}\right)\right) - 2$$

$$g^{-1}(y+2) = g^{-1}g\left(3 + g^{-1}\left(\frac{2}{3x}\right)\right)$$

$$g\left(g^{-1}(y+2) - 3\right) = g\left(g^{-1}\left(\frac{2}{3x}\right)\right)$$

$$\frac{3}{2}g\left(g^{-1}(y+2) - 3\right) = \frac{1}{x}$$

$$\Rightarrow f^{-1}(x) = \frac{2}{3g\left(g^{-1}(y+2) - 3\right)}$$

$$f(x) = 5 - (4 - 2x)^2$$

Will complete Friday

$$f(x) = 3g\left(\frac{x}{5} - 1\right), \qquad g \text{ is even}$$

$$f(x) = h(x) \cdot g(x)$$

Practice Problems: 1.4 page 28 – 31 # 1-10, 12-15, 19-21, C1, C2

Building Understanding 2