The Limit and Limit Computations

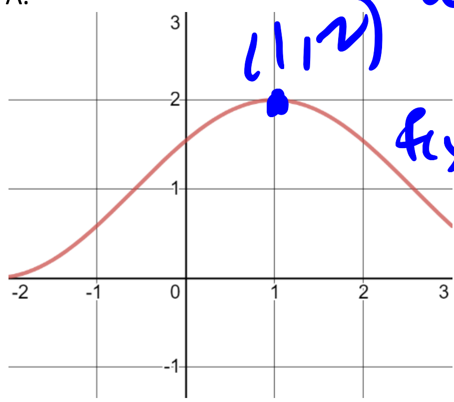
Goal: • Can describe a limit as a shrinking ball. • Can determine a limit from a graph and using a table of values. • Can determine a limit algebraically by factoring and using the conjugate.
Terminology: • Limit • Indeterminate form

Using the graphs and table of values. Describe the behaviour around $x = 1$.

consider both sides of 1 ie/ 0.999

what f(x) does

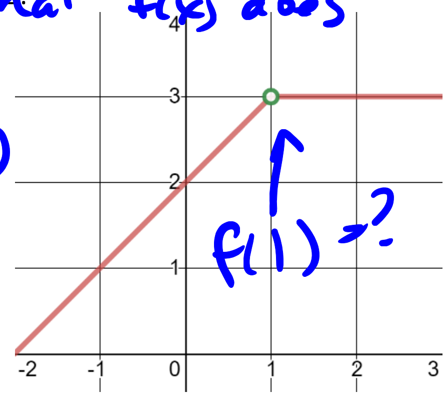
A.



(1, 2)

f(x)

B.



$f(1) = ?$

C.

x	$f(x) = y$
0	7
0.9	7.5
0.95	7.86
0.9999	7.9648
1	9
1.0001	9.0431
1.05	9.14
1.1	9.2
2	10

Your description $\cos(x-1)+1 = f(x)$ $\lim_{x\to 1} \cos(x-1)+1 = \cos(0)+1$ $= 2$		
Class description as $x \to 1$ (both sides) $f(x) \to 2$ $f(1) = 2$	as $x \to 1$ (both sides) $f(x) \to 3$ rate we approach 3 changes	$f(x) \to 9$ from as x decreases to 1 $f(x) \to 8$ as x increases to 1

What you are describing is the basic definition of the limit. That is:

$$\lim_{x\to 1} f(x) = L$$

← The value f(x) gets close to

(both sides)

One way to think of the limit

$$\lim_{x \to c} f(x) = L$$

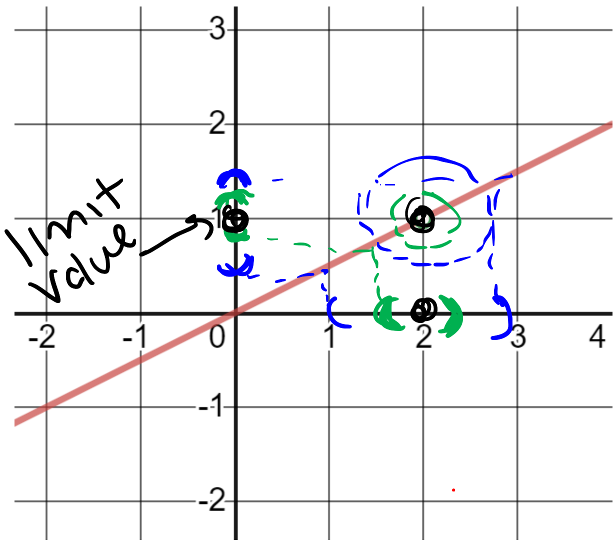
As the idea that **small changes in x around the value c will result in small changes in y around the value L**. In fact, we define the limit using this idea:

Definition: We say $\lim_{x \to c} f(x) = L$ if when we pick values close enough to c (in the domain of f) that all of the values of $f(x)$ will be close to L **with the possible exception of the value of f at $x = c$**

Consider the function $f(x) = \frac{x^2-2x}{2x-4}$ as we let $x \to 2$

$\lim_{x \to 2} f(x) = 1$

$\frac{5}{20}$



Definition: We say that that $f(2) = \frac{0}{0}$ which is **indeterminate** because

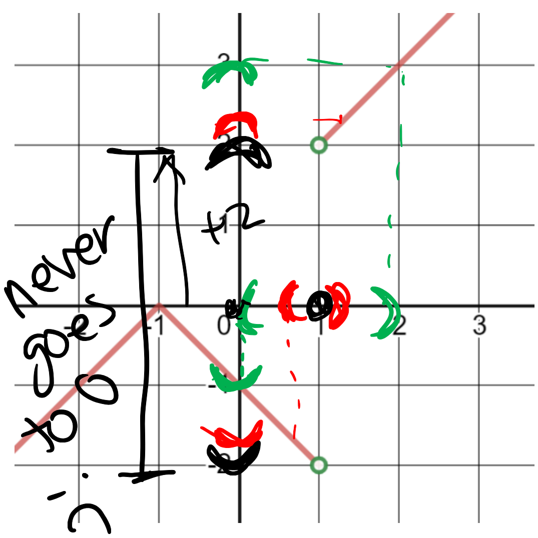
$n = \frac{1}{0}$ is undefined

$n \cdot 0 = 1$ no $n \in \mathbb{R}$ exists!

$m = \frac{0}{0}$

$m \cdot 0 = 0$ every $m \in \mathbb{R}$ works!

Compare this to the function $g(x) = \frac{|x^2-1|}{x-1}$ as we let $x \to 1$



Again, we have that $g(1) = \frac{0}{0}$

as $x \to 1$ and $\Delta x \to 0$

we have $\Delta y \geq 2$

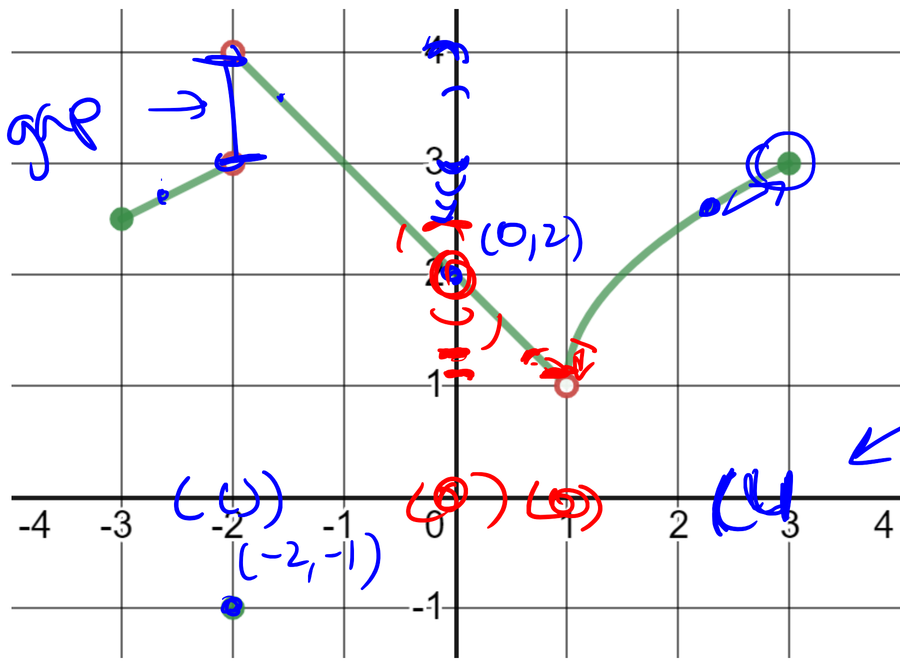
so $\lim_{x \to 1} g(x) = \text{DNE}$

Does NOT exist

When we use the graph to determine a limit, we can imagine a ball centered around the limit point that is getting smaller and smaller around the limit point but still includes other parts of the function.

If we can make a ball arbitrarily small we say the limit exists, but if we can shrink it so the ball contains no other parts of the function (on either side), then the limit does not exist as this would mean a large change in y relative to a small change in x.

Practice: Given the function $f(x)$ below determine the following limits. Note that the domain is $x \in [-3,3]$, $x \neq 1$



f does not exist right of 3

a. $f(-2)$ = -1

b. $f(0)$ = 2

c. $f(1)$ = undefined b/c $1 \notin$ domain

d. $\lim\limits_{x\to -2} f(x)$ = DNE

f(x) does not approach one # from both sides

e. $\lim\limits_{x\to 0} f(x) = 2$

f. $\lim\limits_{x\to 1} f(x) = 1$

☆ $f(x) \to 1$ from above

g. For what values of c is $\lim\limits_{x\to c} f(x) \in \mathbb{R}$

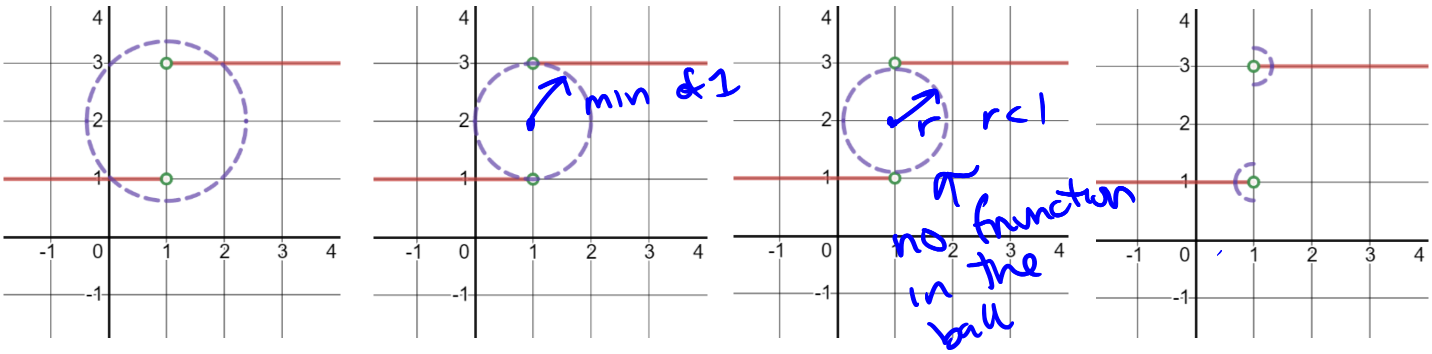
$c \neq -2$

h. For what values of c does $\lim\limits_{x\to c} f(x) = f(c)$

$c \neq -2, 1$

i. $\lim\limits_{x\to 3} f(x) = 3$

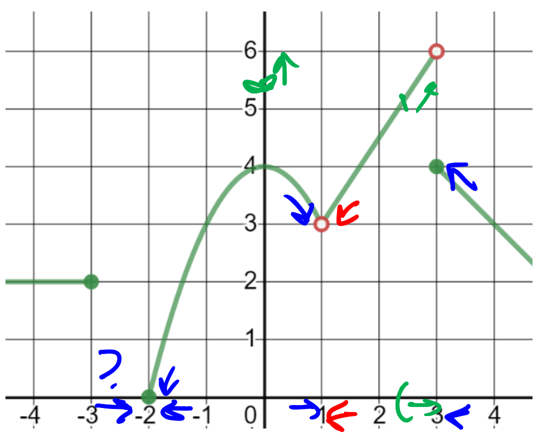
When we talk about the limit as the point an arbitrary sized ball approaches as it shrinks, weird things happen when there is a jump and the ball cannot get any smaller without leaving some of the function behind. We can, however, talk about half balls getting close on either side as $x \to c$



In general, the limit only exists if and only if The left and right hand limits are the same

$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$
right left

Practice: Given the graph of f, determine the following limits. Note the domain of f is $x \le -3$ or $x \ge -2$ and $x \ne 1$



a. $\lim_{x \to 1^-} f(x) = 3$

b. $\lim_{x \to 1^+} f(x) = 3$

c. $\lim_{x \to 1} f(x) = 3$

from the top

d. $\lim_{x \to 3^-} f(x) = 6$

left

e. $\lim_{x \to 3^+} f(x) = 4$

f. $\lim_{x \to 3} f(x) = \text{DNE}$

b/c $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

g. $\lim_{x \to -2^-} f(x)$

undefined

out of domain

h. $\lim_{x \to -2} f(x) = 0$

i. $\lim_{x \to 1} f(f(x))$

$= f\left(\lim_{x \to 1} f(x)\right)$ what side does this approach

$= \lim_{x \to 3^+} f(x) = 4$

When the limit exists, we can extend this to other similar functions using limit properties. We start with the assumptions that:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M$$

Where $L, M \in \mathbb{R}$ (they exist and are some number), then:

1. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
2. $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$
3. $\lim_{x\to c} k\cdot f(x) = k\cdot \lim_{x\to c} f(x) = k\cdot L$
4. $\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = L\cdot M$
5. $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}, \quad M \neq 0$
6. $\lim_{x\to c} g(f(x)) = g\left(\lim_{x\to c} f(x)\right) = g(L)$ where g is a "regular" function whose domain includes L **(more on this tomorrow)

$\lim_{x\to 0} \sin(x+1) = \sin\left(\lim_{x\to 0}(x+1)\right)$

continuous

The proof of these requires a more precise definition of the limit, but we should be able to justify these to when we think of the limit as an approximation to the value of the function.

$$\boxed{\lim_{x\to c} f(x) \approx f(c)}$$

So, all these arithmetic operations should hold when we combine approximations (assuming the approximation is good, ie. The limit exists)

We can also compute limits without their graph or a table by using algebraic operations. Namely factoring and using the conjugate.

Consider the limit:

$$\lim_{x\to 2}\frac{x^2-x-2}{x^2-5x+6}$$

$p(x)$ $q(x)$

$p(2)=0$ $q(2)=0$

$\frac{4-2-2}{4-10+6} = \frac{0}{0} = ?$

factor b/c we have polynomials

$(x-2)$ is a factor of p and q

$$\lim_{x\to 2}\frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-3)} \qquad x \neq 2$$

$$= \lim_{x\to 2}\frac{x+1}{x-3} = \frac{3}{-1} = -3$$

done :)

Evaluate the limit

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1} = \frac{\sqrt{4}-2}{1-1} = \frac{0}{0} \quad ???$$

Conjugate of A+B is A−B

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3}+2)}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Practice: Evaluate the following limits.

$$\frac{\frac{a}{c}+\frac{b}{d}}{e+f}$$

$$\lim_{x\to -2}\frac{\frac{1}{x-2}+\frac{1}{4}}{x+2}$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

$$= \lim_{x\to -2}\frac{4+x-2}{(x+2)4(x-2)}$$

$$= \lim_{x\to -2}\frac{(\cancel{x+2})}{(\cancel{x+2})4(x-2)}$$

$$= \frac{1}{4(-4)} = -\frac{1}{16}$$

$$\lim_{x\to 3}\frac{\sqrt{x-2}-1}{\sqrt{x+6}-3}\cdot\frac{\sqrt{x+6}+3}{''}\cdot\frac{\sqrt{x-2}+1}{''}$$

$$= \lim_{x\to 3}\frac{(\sqrt{x+6}+3)(x-2-1)}{(x+6-9)(\sqrt{x-2}+1)}$$

$$= \lim_{x\to 3}\frac{(\sqrt{x+6}+3)\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x-2}+1)}$$

$$= \frac{\sqrt{9}+3}{\sqrt{1}+1} = \frac{6}{2} = 3$$

Practice Problems: 2.1: # 1-16 (select), 21-25, 33-36, 43, 45-48 (select)	
Textbook Readings: Page 55-61	
Workbook Practice: Page 37-49	
Next Day: Squeeze Theorem and Infinite Limits	