Squeeze Theorem and Limits with Infinity

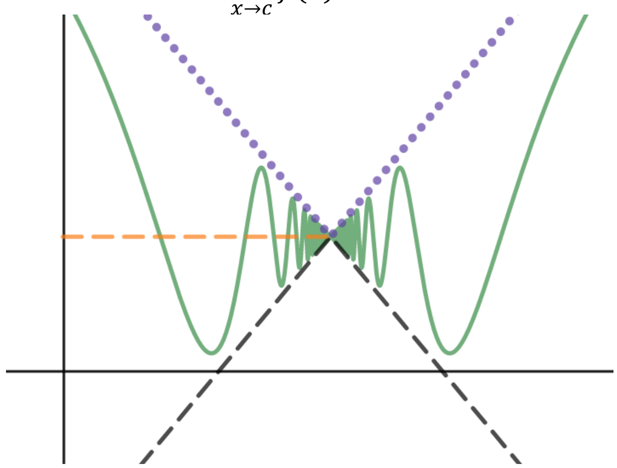
Goal:
- Can use squeeze theorem to determine the value of limits
- Can use the limit of $\frac{\sin x}{x}$ and $\frac{\cos x-1}{x}$ to evaluate certain limits
- Can determine the limit as $x \to \infty$ using substitution, graphs, and squeeze theorem
- Can give an interpretation of a limit being ∞

Terminology:
- Squeeze Theorem

Theorem: Let (a, b) be some interval such that $g(x) \leq f(x) \leq h(x)\ \forall\ x \in (a, b), x \neq c$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

Then

$$\lim_{x\to c} f(x) = L$$

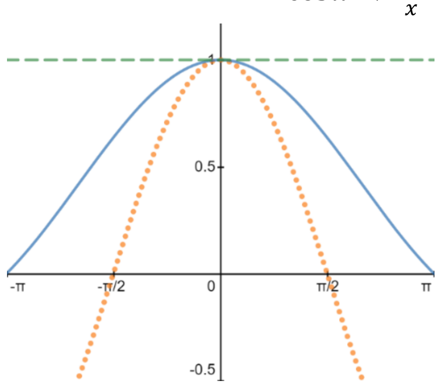


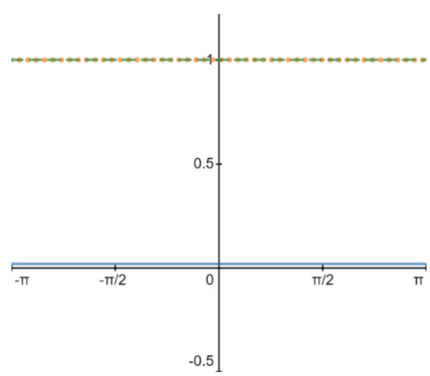
This is the **Squeeze Theorem**.

Consider $f(x) = \frac{\sin x}{x}$ we need to squeeze it between two "nice functions" (that we can find the limit of easily).

$$\cos x < \frac{\sin x}{x} < 1 \text{ provided } x \in (-\pi, \pi)$$

WARNING!!





This limit is central to calculus regarding trig functions and is something we will look to use when evaluating trig limits

Now that we know the $\lim_{x\to 0}\frac{\sin x}{x} = 1$ we can use that limit to find other limits. For example, determine the following.

$$\lim_{x\to 0}\frac{\tan 2x}{x}$$

Practice: Determine

a. $\lim_{x\to 0}\frac{3x}{\sin x}$

b. $\lim_{x\to 0}\frac{\sin x^2}{x}$

c. $\lim_{x\to 0}\frac{x^2+5x-\sin x}{x}$

d. $\lim_{x\to 1}\frac{\sin(x-1)}{x^2-3x+2}$

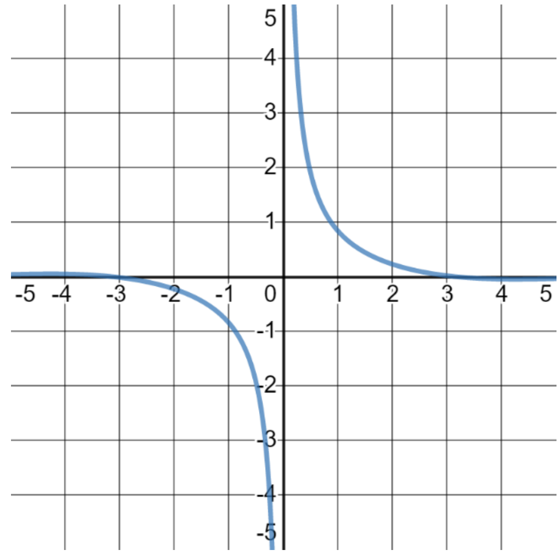
e. $\lim_{x\to 0}\frac{1-\cos x}{x}$

f. $\lim_{x\to \pi}\frac{\sin x}{x-\pi}$

In some cases, such as

$$\lim_{x\to 0}\frac{\sin x}{x^2}$$

The limit does not exist as the indeterminate of $\frac{0}{0}$ is turned into $\frac{0}{0}\cdot\frac{1}{0}$



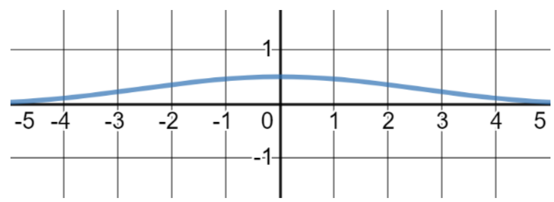
Whenever we have a limit

$$\lim_{x\to c}\frac{f(x)}{g(x)}\to\frac{n}{0}, n\neq 0$$

We have the possibility of have a **vertical asymptote** and we should check what happens on both sides of the limit to see if the function blows up or stays

Counterexample: Consider the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$



Something amazing…

Definition: If when we pick values of x close enough to c (in the domain of f) that all the values of $f(x)$ can be made arbitrary large we say that

$$\lim_{x\to c} f(x)=\infty$$

We can define the right and left-hand limit to be infinity in the same way.

So, in the case of the above

Practice: Find all the values of x where the function has a vertical asymptote. Do the left and right-hand limits equal each other?

a. $\frac{x^2 - 3x + 2}{x^2 - 3x - 4}$

b. $\frac{x^2 + x - 2}{x^3 + 2x^2}$

c. $\frac{x}{1 - \cos x}$

d. $\ln\left(\frac{x - 1}{x}\right)$

If we're comfortable letting $f(x) \to \infty$ as $x \to c$ we should be motivated to consider the case where $f(x) \to L$ as $x \to \infty$

Discussion: How would you define this limit

$$\lim_{x \to \infty} f(x) = L$$

And what would be the interpretation of it on a graph?

When determining these limits, it is necessary to consider the rate of growth of certain classes of functions

- Exponentials: $s(x) = e^x$
- Polynomials and power functions (power > 1): $a(x) = x^N, N > 1$ something like x^2
- Lines: $b(x) = x$
- Radicals and power functions (power < 1): $c(x) = x^n, 0 < n < 1$ something like $\sqrt{x}$
- Logarithms: $d(x) = \ln x$
- Functions that don't grow unbounded: $y_1 = k;\ \ y_2 = \sin x\,;\ \ y_3 = e^{-x};\ \ y_4 = \frac{1}{x}$

We get three cases. Assume that $F(x)$ grows faster than $f(x)$ and $g(x)$ is in the same class of functions as f and we have that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

Case 1: $\lim_{x\to\infty} \frac{f(x)}{g(x)}$

Case 2: $\lim_{x\to\infty} \frac{f(x)}{F(x)}$

Case 3: $\lim_{x\to\infty} \frac{F(x)}{f(x)}$

Practice: Determine the following limits

a. $\lim_{x\to\infty} \frac{\sin x}{x}$

b. $\lim_{x\to\infty} \frac{3x^2 + e^{-x}}{-2x^2}$

c. $\lim_{x\to\infty} \frac{2 + \sqrt{x}}{\ln^2 x}$

d. $\lim_{x\to\infty} \frac{x \cdot \ln x + x}{x^2 + \ln x}$

e. $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x}\right)$

f. $\lim_{x\to 0^+} \sin\frac{1}{x}$

Practice Problems: 2.1: # 26-30, 53-56 2.2: # 1-16 (select), 23-28 (select), 43-46, 49, 50, 52
Textbook Readings: Page 61, 65-68, 71
Workbook Practice: Page 50-59, 69-79
Next Day: Continuity