Squeeze Theorem and Limits with Infinity

Goal:
- Can use squeeze theorem to determine the value of limits
- Can use the limit of $\frac{\sin x}{x}$ and $\frac{\cos x - 1}{x}$ to evaluate certain limits
- Can determine the limit as $x \to \infty$ using substitution, graphs, and squeeze theorem
- Can give an interpretation of a limit being ∞

Terminology:
- Squeeze Theorem

what I want

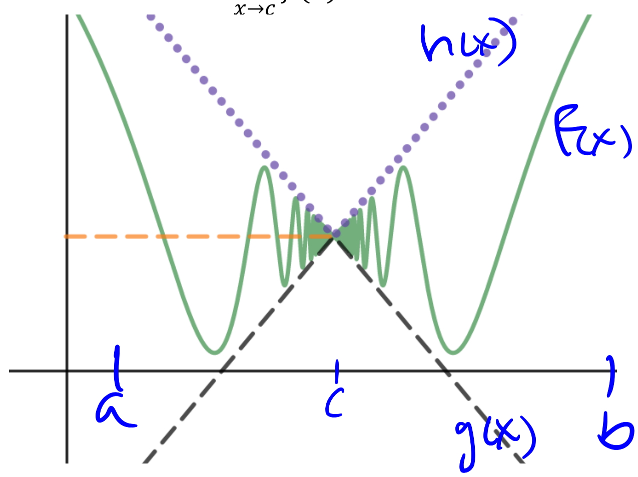
for all

Theorem: Let (a, b) be some interval such that $g(x) \leq f(x) \leq h(x)\ \forall\, x \in (a, b)$, $x \neq c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

basic easy

Then

$$\lim_{x \to c} f(x) = L$$



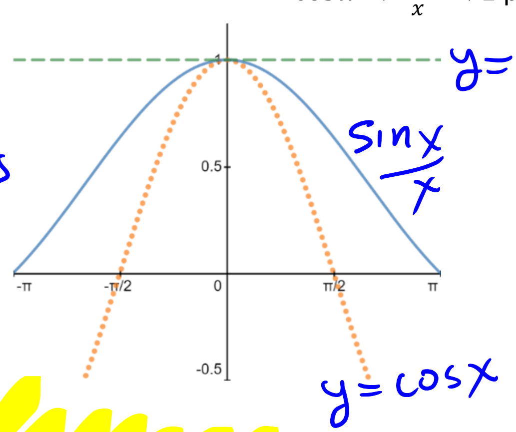
This is the **Squeeze Theorem**.

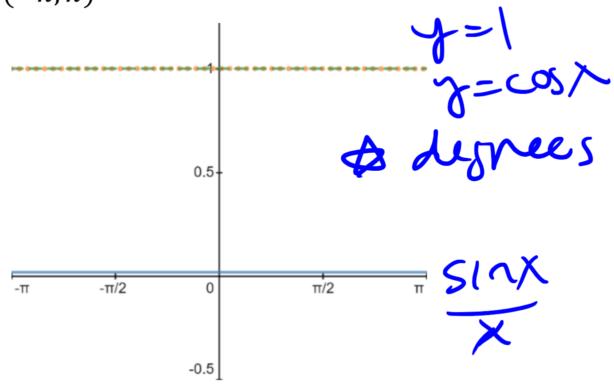
Consider $f(x) = \frac{\sin x}{x}$ we need to squeeze it between two "nice functions" (that we can find the limit of easily).

$$\cos x < \frac{\sin x}{x} < 1 \text{ provided } x \in (-\pi, \pi)$$

WARNING!!

radians





$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0\,(\text{deg})} \frac{\sin x}{x} = \frac{\pi}{180}$$

This limit is central to calculus regarding trig functions and is something we will look to use when evaluating trig limits

Now that we know the $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ we can use that limit to find other limits. For example, determine the following.

$$\lim_{x\to 0} \frac{\tan 2x}{x}$$

let $2x = u$

$$= \lim_{x\to 0} \frac{2\sin 2x}{2x\cos 2x} = 2\lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{1}{\cos 2x}$$

$$2\lim_{u\to 0} \underbrace{\frac{\sin u}{u}}_{1} \cdot \frac{1}{\cos u} = 2 \cdot 1 \cdot \frac{1}{1}$$

$$= 2$$

Practice: Determine

a. $\lim_{x\to 0} \frac{3x}{\sin x}$

$\frac{\sin x}{x} \to 1$

$\frac{x}{\sin x} \to 1$

$$3\lim_{x\to 0} \frac{x}{\sin x}$$

$$3 \cdot 1 = 3$$

b. $\lim_{x\to 0} \frac{\sin x^2}{x^2} \cdot x$

$$\lim_{x\to 0} \frac{\sin x^2}{x^2} \cdot \lim_{x\to 0} x$$

$$\lim_{u\to 0} \frac{\sin u}{u} \cdot \lim_{x\to 0} x = 1 \cdot 0 = 0$$

c. $\lim_{x\to 0} \frac{x^2 + 5x - \sin x}{x}$

$$\lim_{x\to 0} \left(\frac{x^2+5x}{x} - \frac{\sin x}{x} \right)$$

$$\lim_{x\to 0} x+5 - \lim_{x\to 0} \frac{\sin x}{x}$$

$$5 - 1 = 4$$

d. $\lim_{x\to 1} \frac{\sin(x-1)}{x^2 - 3x + 2}$

$$\lim_{x\to 1} \frac{\sin(x-1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{\sin(x-1)}{x-1} \cdot \lim_{x\to 1} \frac{1}{x-2}$$

$$= 1 \cdot \frac{1}{-1} = -1$$

e. $\lim_{x\to 0} \frac{1 - \cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$

$$\lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$\frac{\sin x}{x} \to 1$, $\frac{\sin x}{1+\cos x} \to 0$

$$= 0$$

f. $\lim_{x\to \pi} \frac{\sin x}{x - \pi}$

$x = u + \pi$

$u = x - \pi$

as $x \to \pi$

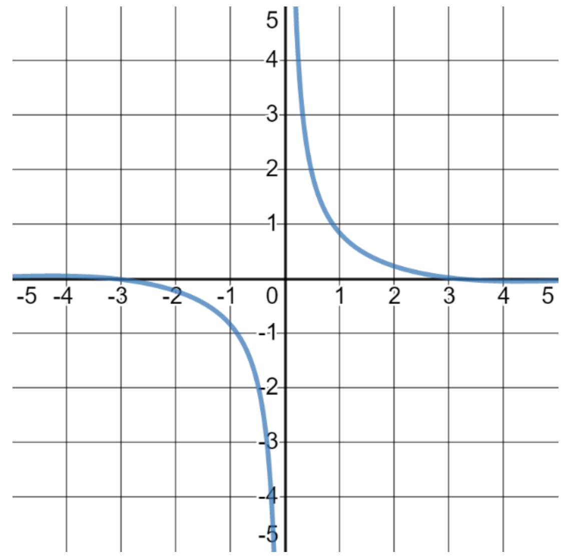
$u \to 0$

$$\lim_{u\to 0} \frac{\sin(u+\pi)}{u}$$

$$\lim_{u\to 0} \frac{-\sin u}{u} = -1$$

In some cases, such as

$$\lim_{x\to 0}\frac{\sin x}{x^2} \qquad \frac{0}{0}$$



The limit does not exist as the indeterminate of $\frac{0}{0}$ is turned into $\frac{0}{0}\cdot\frac{1}{0}$

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{x} \quad \leftarrow \text{problem}$$

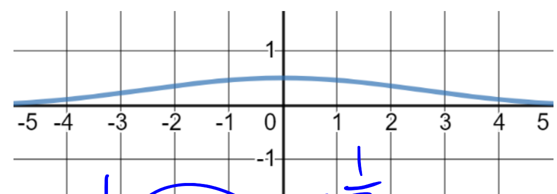
Whenever we have a limit

$$\lim_{x\to c}\frac{f(x)}{g(x)}\to\frac{n}{0}, n\neq 0$$

We have the possibility of have a **vertical asymptote** and we should check what happens on both sides of the limit to see if the function blows up or stays

Counterexample: Consider the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\overbrace{\frac{1-\cos x}{x}}^{0}\cdot\frac{1}{x}$$



$$\lim_{x\to 0}\frac{\sin^2 x}{x^2\cdot(1+\cos x)} = \lim_{x\to 0}\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{1}{1+\cos x}}_{\to \frac{1}{2}} = \frac{1}{2}$$

Something amazing…

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \frac{1-\cos x}{x^2}\approx\frac{1}{2}$$

$\Rightarrow$ for small x $\sin x \approx x$ $\qquad \Rightarrow \cos x \approx 1-\frac{1}{2}x^2$

Definition: If when we pick values of x close enough to c (in the domain of f) that all the values of $f(x)$ can be made arbitrary large we say that

it doesn't matter

$$\lim_{x\to c} f(x) = \infty \qquad \frac{\sin(0.01)}{0.01^2} = -10^6$$

We can define the right and left-hand limit to be infinity in the same way.

So, in the case of the above

$$\lim_{x\to 0^+}\frac{\sin x}{x^2} = \infty \qquad \lim_{x\to 0^-}\frac{\sin x}{x^2} = -\infty$$

Practice: Find all the values of x where the function has a vertical asymptote. Do the left and right-hand limits equal each other?

a. $y = \frac{x^2-3x+2}{x^2-3x-4} = \frac{(x-2)(x-1)}{(x-4)(x+1)}$

$x = 4, -1$

$\lim_{x\to 4^+} y = +\infty$ $\lim_{x\to -1^+} y = \infty$

$\lim_{x\to 4^-} y = -\infty$ $\lim_{x\to -1^-} y = \infty$

b. $\frac{x^2+x-2}{x^3+2x^2} = \frac{(x+2)(x-1)}{x^2(x+2)} = y$

$x = 0$

$\lim_{x\to 0^+} y = -\infty$

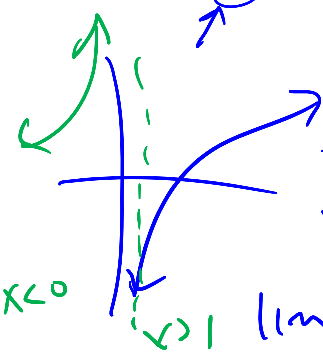
$\lim_{x\to 0^-} y = -\infty$

c. $\frac{x}{1-\cos x}$

$\cos x = 1$ $x = 2\pi n, n\in\mathbb{Z}$

d. $\ln\left(\frac{x-1}{x}\right) = y$

$e^y = \frac{x-1}{x}$



$\ln x, x>0$

$x=1, x>1$

$x<0$

$x>1$ $\lim_{x\to 1^+} \ln\left(\frac{x-1}{x}\right) = -\infty$

$\ln\left(\frac{+0.999}{+0.001}\right)$ ∞

$\lim_{x\to 0^-} \ln\left(\frac{x-1}{x}\right) = +\infty$

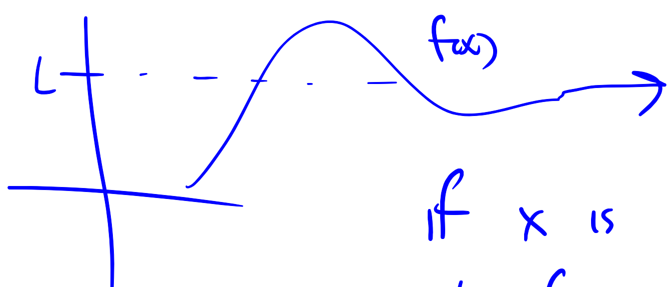
If we're comfortable letting $f(x) \to \infty$ as $x \to c$ we should be motivated to consider the case where $f(x) \to L$ as $x \to \infty$

Discussion: How would you define this limit

$$\lim_{x\to\infty} f(x) = L$$

And what would be the interpretation of it on a graph?

horizontal asymptote



if x is arbitrarily large then $\underbrace{\text{all } f(x) \text{ are really close to } L}_{\text{then } \Delta y \text{ is very small}}$

When determining these limits, it is necessary to consider the rate of growth of certain classes of functions

Fast

$F(x)$

- Exponentials: $s(x) = e^x$

$g(x)=x^3$

$\leftarrow h(x)$

$f(x)=x^2$

- Polynomials and power functions (power > 1): $a(x) = x^N, N > 1$ something like x^2
- Lines: $b(x) = x$
- Radicals and power functions (power < 1): $c(x) = x^n, 0 < n < 1$ something like $\sqrt{x}$
- Logarithms: $d(x) = \ln x$ $\qquad \ln(t) < 20$
- Functions that don't grow unbounded: $y_1 = k$; $y_2 = \sin x$; $y_3 = e^{-x}$; $y_4 = \frac{1}{x}$

Slow

We get three cases. Assume that $F(x)$ grows faster than $f(x)$ and $g(x)$ is in the same class of functions as f and we have that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

$g(x) = 2x^2$

Case 1: $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ $= \lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$

x^2 vs x^3

too different to compare

$= \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$

x^2 vs $2x^2$

$x^2 \approx \frac{1}{2} 2x^2$ for large x

Case 2: $\lim_{x\to\infty} \frac{f(x)}{F(x)}$ $= 0$

$F(x) > h(x) \Leftrightarrow \frac{1}{F(x)} < \frac{1}{h(x)}$

$0 < \frac{f(x)}{F(x)} < \frac{f(x)}{h(x)} \to 0$

Case 3: $\lim_{x\to\infty} \frac{F(x)}{f(x)}$ $= \infty$

$h > f$

$\frac{F(x)}{f(x)} > \frac{h(x)}{f(x)} \to \infty$

$\frac{\text{small}}{\text{Big}} \to 0$ $\qquad \frac{\text{Big}}{\text{small}} \to \infty$

Practice: Determine the following limits

a. $0 < \lim_{x\to\infty} \frac{\sin x}{x} < \frac{1}{x} \to 0$

b. $\lim_{x\to\infty} \frac{3x^2 + \cancel{e^{-x}}}{\cancel{\sin(1/x)} - 2x^2} = -\frac{3}{2}$

c. $\lim_{x\to\infty} \frac{\cancel{2} + \sqrt{x}}{\ln^2 x} = \infty$

d. $\lim_{x\to\infty} \frac{\boxed{x \cdot \ln x} + x}{x^2 + \ln x} = 0$

e. $\lim_{x\to\infty} \left(\sqrt{x} - \sqrt{x^2 - x}\right)$

$\sqrt{x^2 - x} \approx x$

$\cdot\frac{(\sqrt{x} + \sqrt{x^2 - x})}{(\sqrt{x} + \sqrt{x^2 - x})}$

$\lim_{x\to\infty} \frac{x - (x^2 - x)}{\sqrt{x} + \sqrt{x^2 - x}}$

$\lim_{x\to\infty} \frac{-x^2 + 2x}{\sqrt{x} + \sqrt{x^2 - x}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

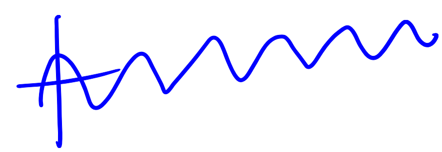
$\lim_{x\to\infty} \frac{-1 + \frac{2}{x^2}}{\sqrt{\frac{x}{x^4}} + \sqrt{\frac{x^2}{x^4} - \frac{x}{x^4}}}$

f. $\lim_{x\to 0^+} \sin\frac{1}{x}$

$\frac{1}{x} = u$

$\lim_{u\to\infty} \sin u$ as $x \to 0^+$, $u \to +\infty$

DNE



$x^2\sqrt{x} = \sqrt{x^4 \cdot x}$

Practice Problems: 2.1: # 26-30, 53-56 2.2: # 1-16 (select), 23-28 (select), 43-46, 49, 50, 52	
Textbook Readings: Page 61, 65-68, 71	
Workbook Practice: Page 50-59, 69-79	
Next Day: Continuity	