Squeeze Theorem and Limits with Infinity

Goal: • Can use squeeze theorem to determine the value of limits • Can use the limit of $\frac{\sin x}{x}$ and $\frac{\cos x - 1}{x}$ to evaluate certain limits • Can determine the limit as $x \to \infty$ using substitution, graphs, and squeeze theorem • Can give an interpretation of a limit being ∞
Terminology: • Squeeze Theorem

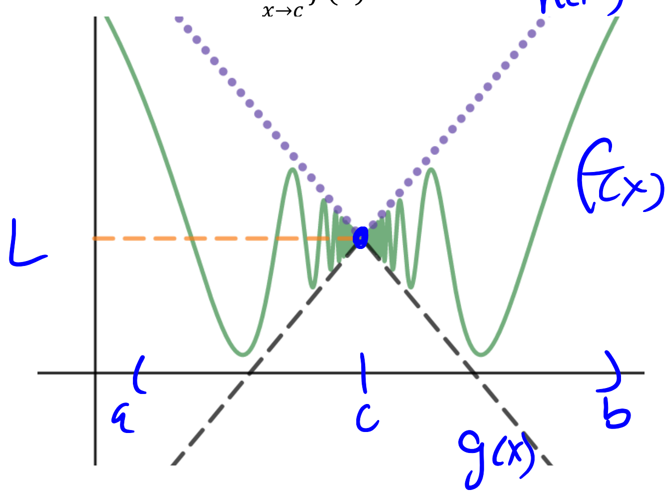
for all/every

Theorem: Let (a, b) be some interval such that $g(x) \leq f(x) \leq h(x)\ \forall\ x \in (a, b), x \neq c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

L L

Then

$$\lim_{x \to c} f(x) = L$$



This is the **Squeeze Theorem**.

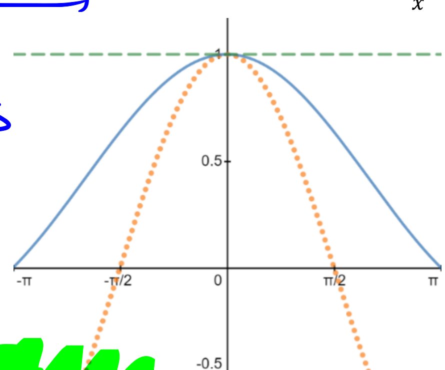
Consider $\boxed{f(x) = \frac{\sin x}{x}}$ we need to squeeze it between two "nice functions" (that we can find the limit of easily).

$$\cos x < \frac{\sin x}{x} < 1 \text{ provided } x \in (-\pi, \pi)$$

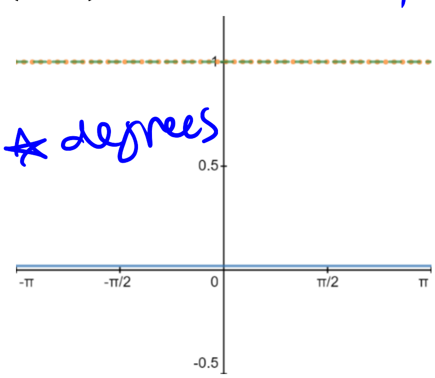
#13 2.1

WARNING!!

* radians



* degrees



$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\frac{x}{\sin x} \to 1$ as $x \to 0$

$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$

This limit is central to calculus regarding trig functions and is something we will look to use when evaluating trig limits

Now that we know the $\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$ we can use that limit to find other limits. For example, determine the following.

$$\lim_{x\to 0}\frac{\tan 2x}{x}$$

$$\lim_{x\to 0}\frac{\sin 2x}{\cos 2x \cdot 2x}\cdot 2 = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}$$

$u = 2x$
as $x\to 0$
$u\to 0$

$$= \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{2}{\cos 2x} \to 2$$

$$= \lim_{u\to 0}\frac{\sin u}{u}\cdot 2 = 1\cdot 2 = 2$$

Practice: Determine

a. $\lim_{x\to 0}\frac{3x}{\sin x} = 3$

b. $\lim_{x\to 0}\frac{\sin x^2}{x}\cdot\frac{x}{x}$

$$= \lim_{x\to 0} x\cdot\frac{\sin(x^2)}{x^2} \to 1$$

$u = x^2$
as $x\to 0$
$u\to 0$

$$= 0\cdot 1 = 0$$

c. $\lim_{x\to 0}\frac{x^2+5x-\sin x}{x} = 4$

$$\lim_{x\to 0}\left(x+5-\frac{\sin x}{x}\right)$$

$$= 5-1 = 4$$

d. $\lim_{x\to 1}\frac{\sin(x-1)}{x^2-3x+2} = -1$

$$\lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x-2)} = \lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{1}{x-2}$$

$$= \lim_{u\to 0}\frac{\sin u}{u}\cdot\frac{1}{-1} = -1$$

e. $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$ $\quad\frac{1+\cos x}{1+\cos x}$

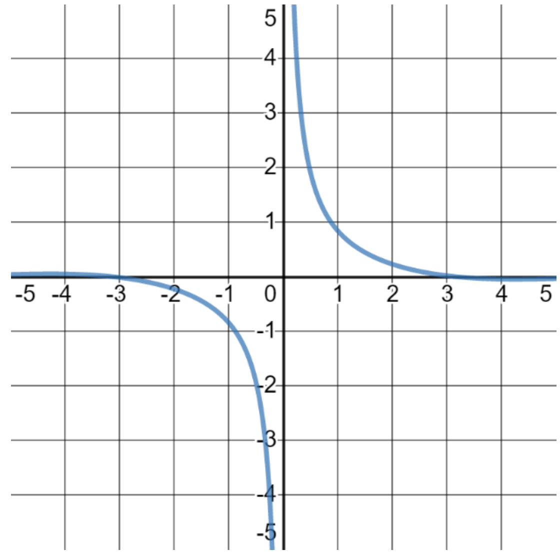
$$\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} = 1\cdot\frac{0}{2} = 0$$

f. $\lim_{x\to\pi}\frac{\sin x}{x-\pi}$ $\quad\frac{\sin(x-\pi)}{x-\pi}$

$$-\lim_{x\to\pi}\frac{-\sin x}{x-\pi} = -\lim_{x\to\pi}\frac{\sin(x-\pi)}{x-\pi}$$

$$= -1$$



In some cases, such as

$$\lim_{x\to 0}\frac{\sin x}{x^2}$$

The limit does not exist as the indeterminate of $\frac{0}{0}$ is turned into $\frac{0}{0}\cdot\frac{1}{0}$

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{x} = \frac{0}{0}\cdot\frac{1}{0}$$

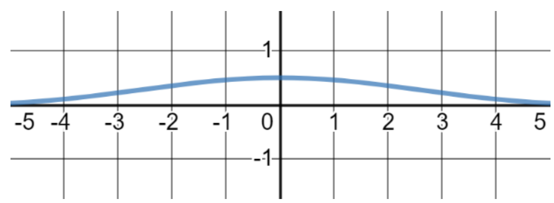
Whenever we have a limit

$$\lim_{x\to c}\frac{f(x)}{g(x)} \to \frac{n}{0}, n \neq 0$$

We have the possibility of have a **vertical asymptote** and we should check what happens on both sides of the limit to see if the function blows up or stays

Counterexample: Consider the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\underbrace{\frac{1-\cos x}{x}}_{\to 0}\cdot\frac{1}{x} \qquad \frac{0}{0}\cdot\frac{1}{0}$$



$$\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\underbrace{\frac{\sin^2 x}{x^2}}_{\to 1}\cdot\lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{2}$$

Something amazing…

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

for $x \approx 0$ $\frac{\sin x}{x} \approx 1$

$\Rightarrow \boxed{\sin x \approx x}$

For $x \approx 0$ $\frac{1-\cos x}{x^2} \approx \frac{1}{2}$

$\cos x \approx 1 - \frac{1}{2}x^2$

Definition: If when we pick values of x close enough to c (in the domain of f) that all the values of $f(x)$ can be made arbitrary large we say that

$$\lim_{x\to c} f(x) = \infty$$

We can define the right and left-hand limit to be infinity in the same way.

So, in the case of the above $\lim_{x\to 0^+}\frac{\sin x}{x^2} = \infty$ $\lim_{x\to 0^-}\frac{\sin x}{x^2} = -\infty$

Practice: Find all the values of x where the function has a vertical asymptote. Do the left and right-hand limits equal each other?

a. $y = \dfrac{x^2 - 3x + 2}{x^2 - 3x - 4} = \dfrac{(x-2)(x-1)}{(x-4)(x+1)}$

$x = 4 \quad x = 1$

$\lim_{x\to 4^+} y = +\infty \qquad \lim_{x \to 4^-} y = -\infty$

b. $\dfrac{x^2 + x - 2}{x^3 + 2x^2} = \dfrac{(x-1)\cancel{(x+2)}}{x^2\cancel{(x+2)}}$

$x = 0$

$\lim_{x\to 0^+} \dfrac{x-1}{x^2} = -\infty$

$\lim_{x\to 0^-} \dfrac{x-1}{x^2} = -\infty$

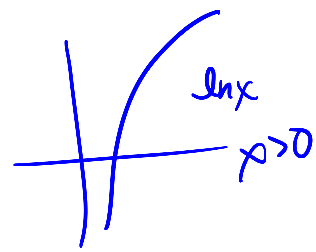
c. $\dfrac{x}{1 - \cos x}$

$\cos x = 1 \Rightarrow x = 2\pi n, \ n \in \mathbb{Z}$

d. $\ln\left(\dfrac{x-1}{x}\right)$

$x = 1 \ \& \ x > 1$

$x - 1 < 0$
$x < 0 \Rightarrow x < 0$



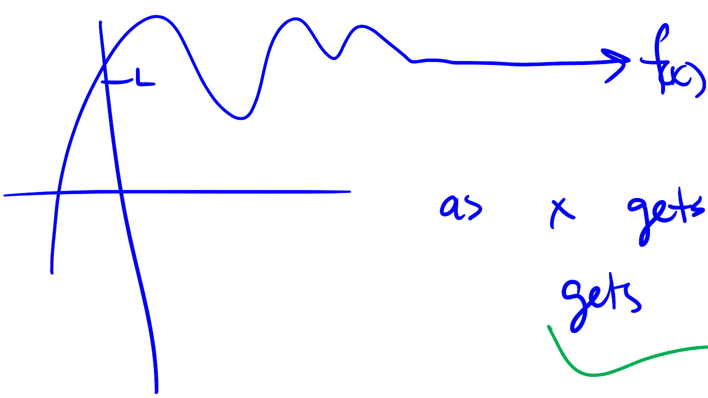
$\lim_{x\to 0} \ln\left(\dfrac{x-1}{x}\right)$

If we're comfortable letting $f(x) \to \infty$ as $x \to c$ we should be motivated to consider the case where $f(x) \to L$ as $x \to \infty$

Discussion: How would you define this limit

$$\lim_{x\to\infty} f(x) = L$$

And what would be the interpretation of it on a graph?



as x gets arbitrarily large then $f(x)$ gets really close to L

The Δy is very small

When determining these limits, it is necessary to consider the rate of growth of certain classes of functions

Fast $F(x)$

- Exponentials: $s(x) = e^x$
- $h(x)$ Polynomials and power functions (power > 1): $a(x) = x^N, N > 1$ something like x^2
- Lines: $b(x) = x$
- $f(x)$, $g(x)$ Radicals and power functions (power < 1): $c(x) = x^n, 0 < n < 1$ something like $\sqrt{x}$
- Logarithms: $d(x) = \ln x$ $f(t) = \ln t < 20$
- Functions that don't grow unbounded: $y_1 = k$; $y_2 = \sin x$; $y_3 = e^{-x}$; $y_4 = \frac{1}{x}$

Slow

We get three cases. Assume that $F(x)$ grows faster than $f(x)$ and $g(x)$ is in the same class of functions as f and we have that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

Case 1: $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ $\frac{\text{Small}}{\text{Big}} \to 0$ $\frac{\text{Big}}{\text{small}} \to \infty$

i) $f(x) = \sqrt{x}$ $g(x) = x$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$f(x) = \sqrt{x}$ $g(x) = x^{\frac{1}{9}}$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x^{1/9}} = \lim_{x\to\infty} x^{1/9} = \infty$$

$f(x) = \sqrt{x}$ $g(x) = 2\sqrt{x}$

$$\boxed{\lim_{x\to\infty} \frac{\sqrt{x}}{2\sqrt{x}} = \frac{1}{2}}$$

Case 2: $\lim_{x\to\infty} \frac{f(x)}{F(x)} = 0$ $h < F \iff \frac{1}{h} > \frac{1}{F}$

$$0 < \frac{f(x)}{F(x)} < \frac{f(x)}{h(x)} \to 0$$

Case 3: $\lim_{x\to\infty} \frac{F(x)}{f(x)} = \infty$

$$\infty > \frac{F}{f} > \frac{h}{f} \to \infty$$

Practice: Determine the following limits

a. $\lim_{x\to\infty}\frac{\sin x}{x}$ small BIG $= 0$

b. $\lim_{x\to\infty}\frac{3x^2+e^{-x}}{-2x^2+\sin\left(\frac{1}{x}\right)} = \frac{3}{-2}$

c. $\lim_{x\to\infty}\frac{2+\sqrt{x}}{\ln^2 x}$ BIG small $= \infty$

d. $\lim_{x\to\infty}\frac{x\ln x + x}{x^2+\ln x} = 0$ small BIG

$2x$ to x

e. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

$$\lim_{x\to\infty}\frac{\cancel{x^2}+x+(\cancel{x^2}+x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

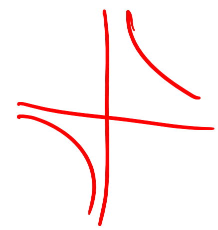
$$\lim_{x\to\infty}\frac{2x/\cancel{x}}{\sqrt{x^2+x}+\sqrt{x^2-x}\,/\cancel{x}} = 1$$

$$\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1$$

$\frac{1}{x}\to 0$, $\frac{1}{x}\to 0$

f. $\lim_{x\to 0^+}\sin\frac{1}{x}$

$u = \frac{1}{x}$

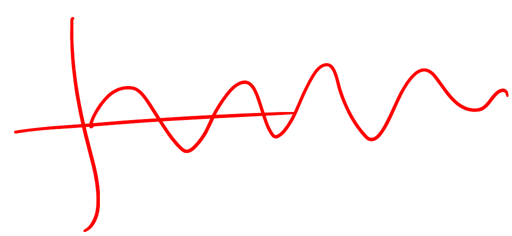


$\lim_{u\to\infty}\sin u$

$x\to 0^+$

$u\to +\infty$

DNE



Practice Problems: 2.1: # 26-30, 53-56 2.2: # 1-16 (select), 23-28 (select), 43-46, 49, 50, 52	
Textbook Readings: Page 61, 65-68, 71	
Workbook Practice: Page 50-59, 69-79	
Next Day: Continuity	