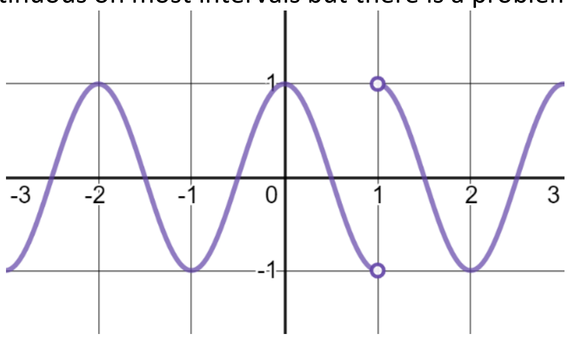
# Continuity and Intermediate Value Theorem

**Goal**:
- Can use the definition of continuity
- Can identify types of discontinuities
- Understands the significance and application of the intermediate value theorem for continuous functions.

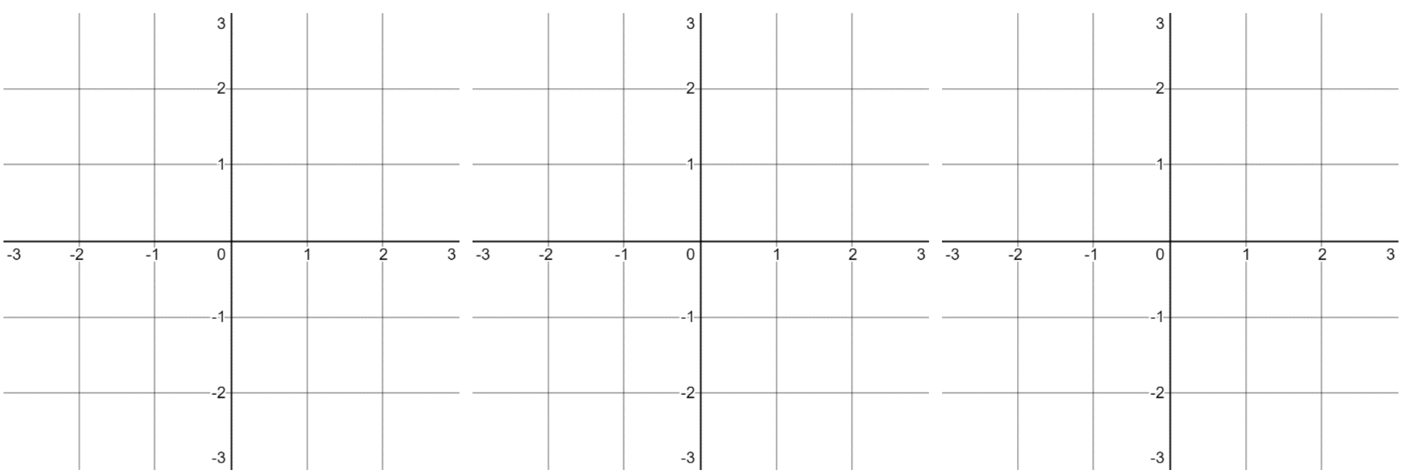
**Terminology**:
- Continuity at a point
- Removeable, jump, essential discontinuities
- Intermediate Value Theorem

**(Weak) Definition**: A function $f$ is continuous on the interval $[a, b]$ if and only if it can be drawn with a line that does not come off the paper.

For example, the graph below is continuous on most intervals but there is a problem at $x = 1$ that we




Draw different ways that a function could fail to be continuous at a point $x = c$. Then using limits, explain why these are not continuous.

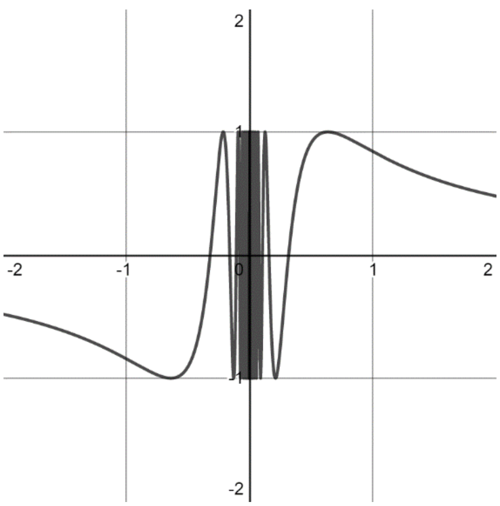


**(Strong) Definition:** A function $f$ is continuous on the interval $[a, b]$ if and only if

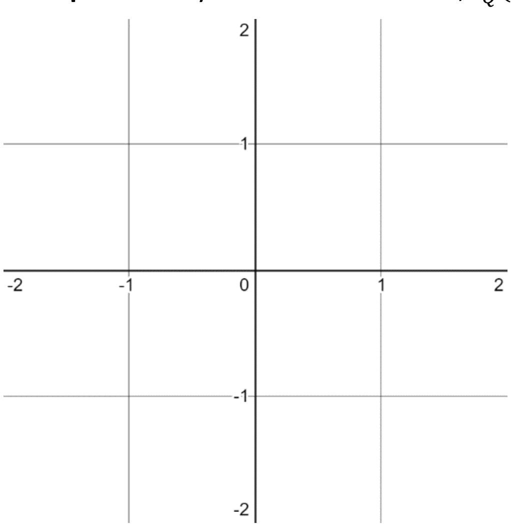
If a function fails the definition of continuity it is said to be **discontinuous**

There are a couple other interesting ways we could be discontinuous:

**Example 1**: Oscillating Discontinuity, $f(x) = \sin\left(\frac{1}{x}\right)$




**Example 2**: Everywhere discontinuous, $I_Q(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$



Arguably the only nice discontinuities are removeable because they can easily be "fixed" through the use of a piecewise function.

For example, we can make the function $f(x) = \frac{\sin x}{x}$ continuous at $x = 0$

**Example**: Make the following functions continuous $\forall\ x \in \mathbb{R}$

a. $f(x) = \dfrac{x^2 + x - 12}{x - 3}$

b. $g(x) = \dfrac{\sec x - 1}{x}$

c. $h(x) = \begin{cases} 2^x, & x < 1 \\ ax - 1, & x \geq 1 \end{cases}$

Find $a$ so $h$ is continuous

d. $k(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x < 0 \\ u(x), & 0 \leq x \leq 2 \\ \dfrac{\sin(\pi x)}{x - 2}, & x > 2 \end{cases}$

Find $u(x)$ so $k$ is continuous

**Property:** If $f$ is continuous at the point $b$ and $\lim_{x\to c} f(x) = b$ then

$$\lim_{x\to c} g(f(x)) = g\left(\lim_{x\to c} f(x)\right) = g(b)$$

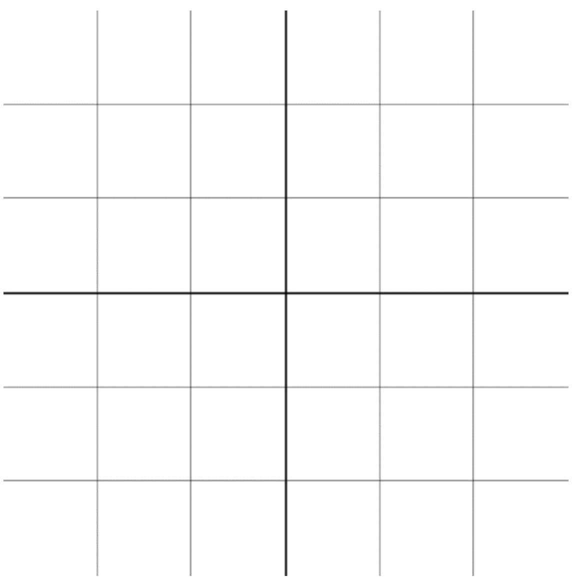
**Property**: If $f$ and $g$ are continuous then $f(g(x))$ is continous.

Prove this!

Continuity is one of the *most important characteristics* of functions and a requirement for many theorems and properties of functions. The first such example you will see is **The Intermediate Value Theorem** which states:

**Theorem:** If $f$ is continuous on the interval $[a, b]$, then it takes on every value between $f(a)$ and $f(b)$.

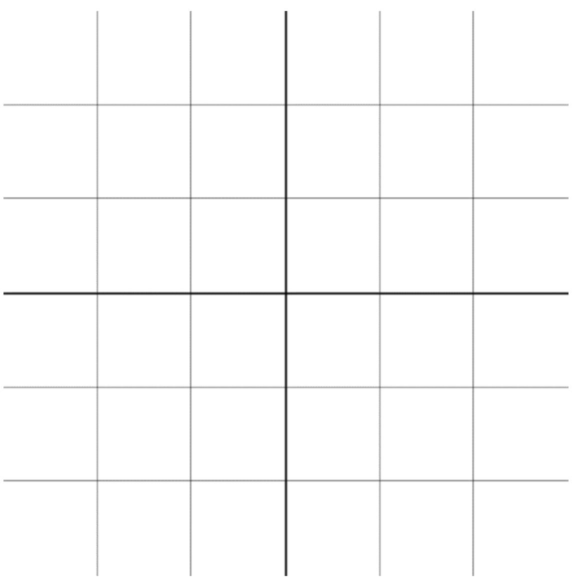
Illustrate this theorem and make an argument why it is true.



Using the theorem we can show that things exist. For example, show that $x^2 + x = \cos x$ has a solution.

**Practice**: Show that $\sin(\cos x) = x^2$ has at least one solution

Illustrate why IVT does not hold when a function is continuous on $(a, b)$



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| **Practice Problems**: 2.3: # 11-18, 19-30 (select), 35-42, 46-49 |
| **Textbook Readings**: Page 73-79 |
| **Workbook Practice**: Page 60-68, 80-81 |
| **Next Day**: Instantaneous rate of change and derivative at a point |