Continuity and Intermediate Value Theorem

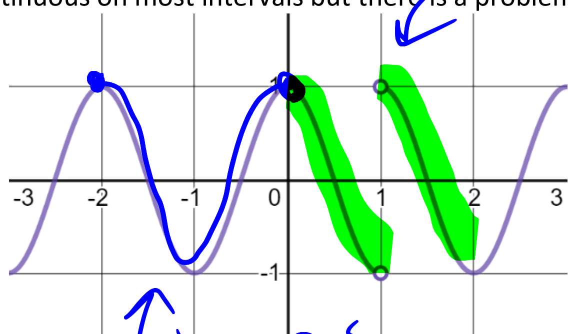
Goal: • Can use the definition of continuity • Can identify types of discontinuities • Understands the significance and application of the intermediate value theorem for continuous functions.
Terminology: • Continuity at a point • Removeable, jump, essential discontinuities • Intermediate Value Theorem

(Weak) Definition: A function f is continuous on the interval $[a, b]$ if and only if it can be drawn with a line that does not come off the paper.

problem @ $x=1$

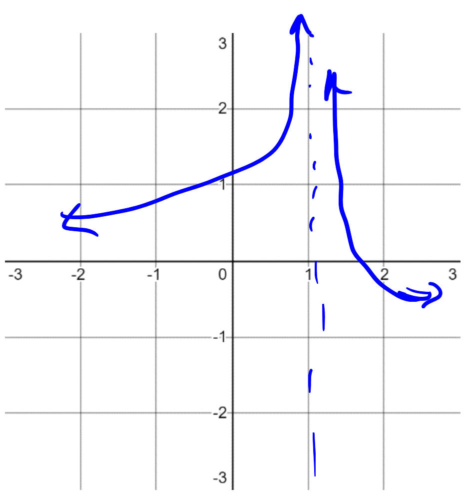
For example, the graph below is continuous on most intervals but there is a problem at $x = 1$ that we

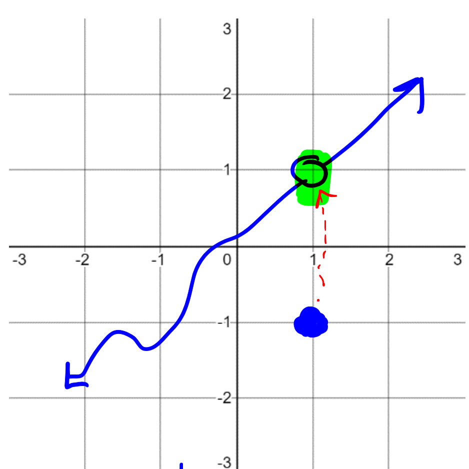
Not continuous

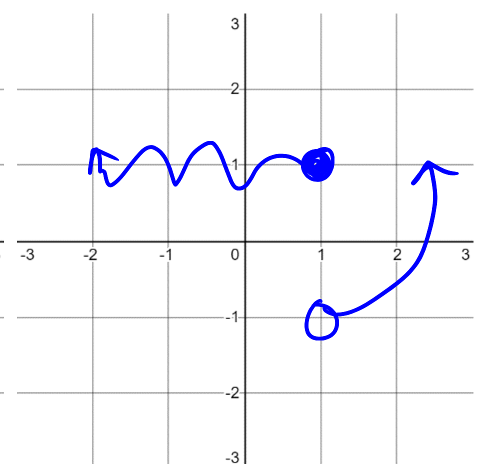


continuous

Draw different ways that a function could fail to be continuous at a point $x = c$. Then using limits, explain why these are not continuous.







asymptote

$\lim_{x \to 1} f(x) = +\infty$

$f(1) = \text{undefined}$

hole / removeable

$\lim_{x \to 1} f(x) = 1$

not the same

$f(1) = -1$

jump

$f(1) = 1$

$\lim_{x \to 1} f(x)$ DNE

(Strong) Definition: A function f is continuous on the interval $[a, b]$ if and only if

$$\lim_{x \to c} f(x) = f(c)$$

must exist

must be defined

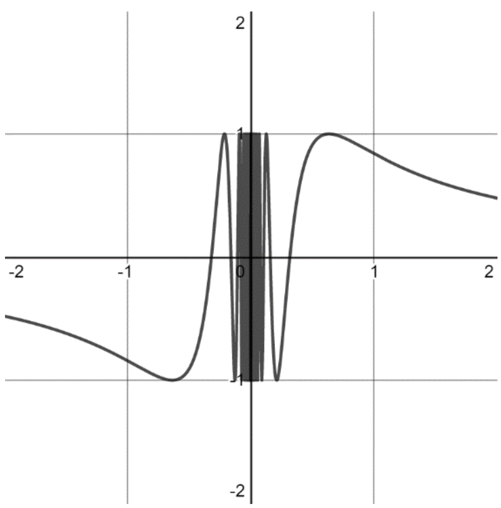
If a function fails the definition of continuity it is said to be **discontinuous**

There are a couple other interesting ways we could be discontinuous:

Example 1: Oscillating Discontinuity, $f(x) = \sin\left(\frac{1}{x}\right)$

$u = \frac{1}{x}$ as $x \to 0$, $u \to \pm\infty$

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = \lim_{u \to \infty} \sin u = \text{DNE}$$



I for indicator

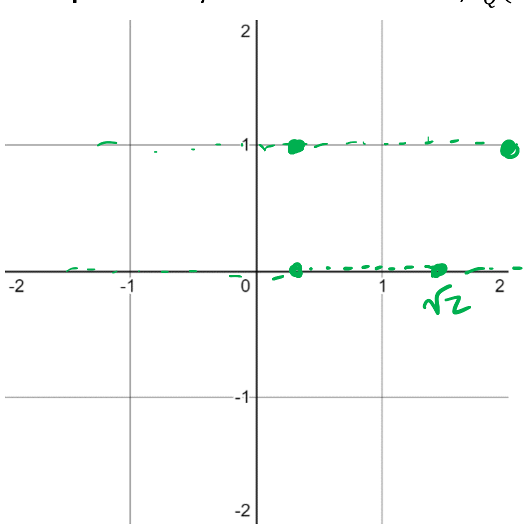
$\mathbb{Q}$ set of rationals

$\frac{a}{b} \in \mathbb{Q}$ if $a, b \in \mathbb{Z}$ Zahlen

Integer

Example 2: Everywhere discontinuous, $I_{\mathbb{Q}}(x) = \begin{cases} 1, & \boxed{x \in \mathbb{Q}} \\ 0, & \boxed{x \in \mathbb{R}\setminus\mathbb{Q}} \end{cases}$

$\mathbb{R}$ minus $\mathbb{Q}$



$I_{\mathbb{Q}}(2) = 1$ $\quad I_{\mathbb{Q}}\left(\frac{1}{\pi}\right) = 0$

$I_{\mathbb{Q}}\left(\frac{1}{3}\right) = 1$

$I_{\mathbb{Q}}(\sqrt{2}) = 0$

Arguably the only nice discontinuities are removeable because they can easily be "fixed" through the use of a piecewise function.

For example, we can make the function $f(x) = \frac{\sin x}{x}$ continuous at $x = 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

fill to match the limit

Example: Make the following functions continuous $\forall\, x \in \mathbb{R}$

a. $f(x) = \dfrac{x^2 + x - 12}{x - 3}$

$$\lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)} = 7$$

$$F(x) = \begin{cases} \frac{x^2+x-12}{x-3}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$

b. $g(x) = \dfrac{\sec x - 1}{x} \cdot \dfrac{\sec x + 1}{\sec x + 1}$

$$\lim_{x\to 0} \frac{\tan^2 x}{x(\sec x + 1)} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{1} \cdot \underbrace{\frac{\sin x}{\cos^2 x(\sec x+1)}}_{0} = 0$$

$$G(x) = \begin{cases} \frac{\sec x - 1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

c. $h(x) = \begin{cases} 2^x, & x < 1 \\ ax - 1, & x \geq 1 \end{cases}$

Find a so h is continuous

$$\lim_{x\to 1^-} 2^x = \lim_{x\to 1^+} ax - 1$$

$$2 = a - 1$$

$$a = 3$$

d. $k(x) = \begin{cases} \dfrac{1-\cos x}{x}, & x < 0 \\ u(x), & 0 \leq x \leq 2 \\ \dfrac{\sin(\pi x)}{x-2}, & x > 2 \end{cases}$

Find $u(x)$ so k is continuous

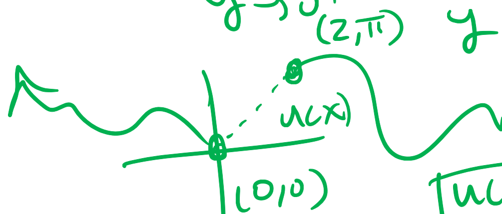
$$\lim_{x\to 0^-} \frac{1-\cos x}{x} = 0 = u(0)$$

$$\lim_{x\to 2^+} \frac{\sin \pi x}{x-2}$$

let $x - 2 = y$ as $x \to 2^+$, $y \to 0^+$

shift by period

$$\lim_{y\to 0^+} \frac{\sin \pi(y+2)}{y} = \lim_{y\to 0} \frac{\sin \pi y}{y} \cdot \frac{\pi}{\pi} = \pi$$



$$\boxed{u(x) = \frac{\pi}{2}x}$$

outside → regular functions

$\sin x$, e^x, $\ln x$, x^2, $\sqrt{x}$

Property: If f g is continuous at the point b and $\lim_{x\to c} f(x) = b$ then

$$\lim_{x\to c} g\big(f(x)\big) = g\left(\lim_{x\to c} f(x)\right) = g(b)$$

proof needs precise def.

Property: If f and g are continuous then $f\big(g(x)\big)$ is continous.

$f+g$ $\quad g\cdot f$ $\quad f/g,\ g>0$

Prove this!

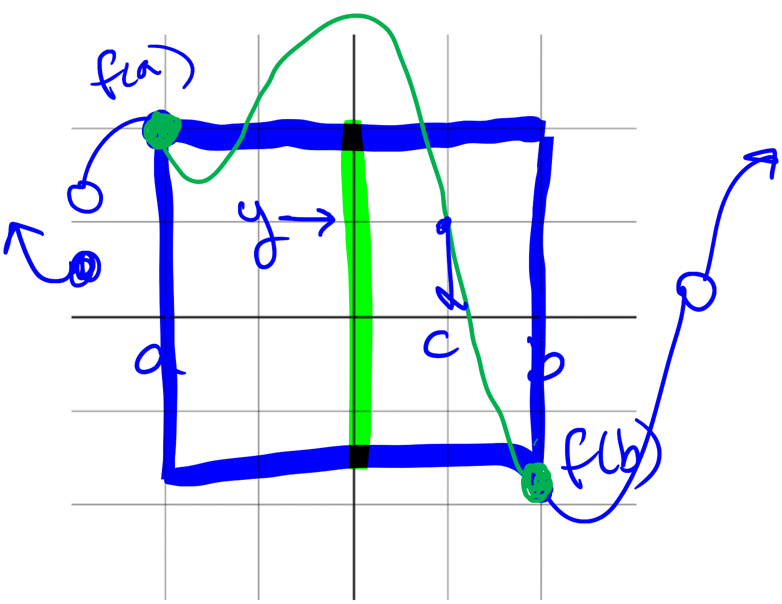
$$\lim_{x\to c}\Big[\underbrace{f}_{\text{cont.}}\big(g(x)\big)\Big] = f\Big(\lim_{x\to c} \underbrace{g(x)}_{\text{cont.}}\Big) = f\big(g(c)\big) \Rightarrow f\big(g(x)\big) \text{ is cont.}$$

Continuity is one of the *most important characteristics* of functions and a requirement for many theorems and properties of functions. The first such example you will see is **The Intermediate Value Theorem** which states:

Theorem: If f is continuous on the interval $[a, b]$, then it takes on every value between $f(a)$ and $f(b)$.

closed

Illustrate this theorem and make an argument why it is true.



for every y value between $f(a)$ and $f(b)$

$\exists$

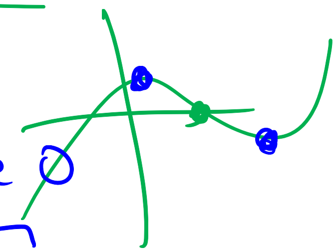
There exists $c \in (a,b)$

such that $f(c) = y$

Using the theorem we can show that things exist. For example, show that $x^2 + x = \cos x$ has a solution.

$f(x) = \underbrace{x^2}_{\text{cont}} + \underbrace{x}_{\text{cont}} - \underbrace{\cos x}_{\text{cont}} \stackrel{TST}{=} 0$ ① set $=0$

is continuous



② find -ve ③ find +ve above 0

$f(0) = -1 < 0$ below 0 $\quad f(1) = 1 + 1 - \cos 1 > 0$, $\exists\, c \in (0,1)$ such that $f(c) = 0$

done :)

Practice: Show that $\sin(\cos x) = x^2$ has at least one solution

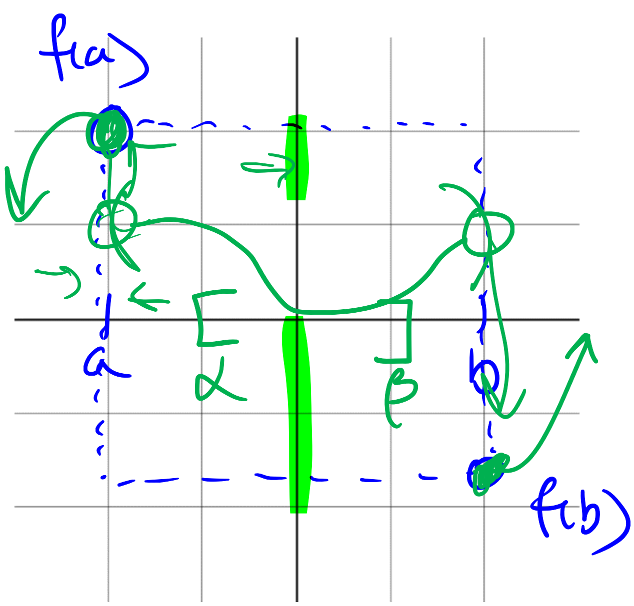
$g(x) = \sin(\cos x) - x^2 \stackrel{TST}{=} 0$

$g(1) = -0.5 < 0$ $\quad \exists\, c \in (0,1)$ s.t $g(c) = 0$ by IVT :)

$g(0) = 0.8 > 0$

o I ~ 10^{-10} m

Illustrate why IVT does not hold when a function is continuous on (a, b) open



* need continuity on $[a, b]$

$L = \lim_{x \to a^+} f(x)$

$R = \lim_{x \to b^-} f(x)$

we will hit every # between L & R on (a,b) not L & R ?

Practice Problems: 2.3: # 11-18, 19-30 (select), 35-42, 46-49	
Textbook Readings: Page 73-79	
Workbook Practice: Page 60-68, 80-81	
Next Day: Instantaneous rate of change and derivative at a point	