# Rates of Change and Derivative

**Goal**:

- Can approximate rate of change from data
- Can determine the precise rate of change of a function using limits.
- Understands the derivative as the slope of the tangent line
- Can use Newton and Leibnitz notation for the derivative
- Understands the difference between $\frac{d}{dx}f(a)$ and $f'(a)$
- Can use calculator to determine the derivative at a point

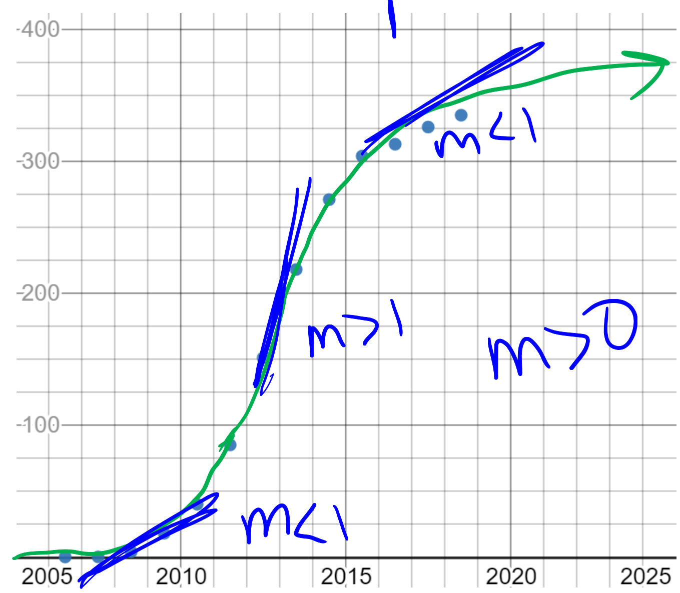
**Terminology**:

- Average rate of change
- Tangent
- Derivative
- $\frac{d}{dx}$ and $f'(x)$

| $t$ | 2006 | 2008 | 2010 | 2011 | 2013 | 2014 | 2016 | 2018 |
|---|---|---|---|---|---|---|---|---|
| $T(t)$ | 0.02 | 4 | 40 | 85 | 218 | 271 | 313 | 335 |

https://www.statista.com/statistics/282087/number-of-monthly-active-twitter-users/

Above are selected values of the time $t$ measured during the middle of the year in June for the number of active Twitter users.




**On the board**:
How has the rate of Twitter's growth changed over the past 15 years?

Give at least two approximations for the rate Twitter was growing in 2012

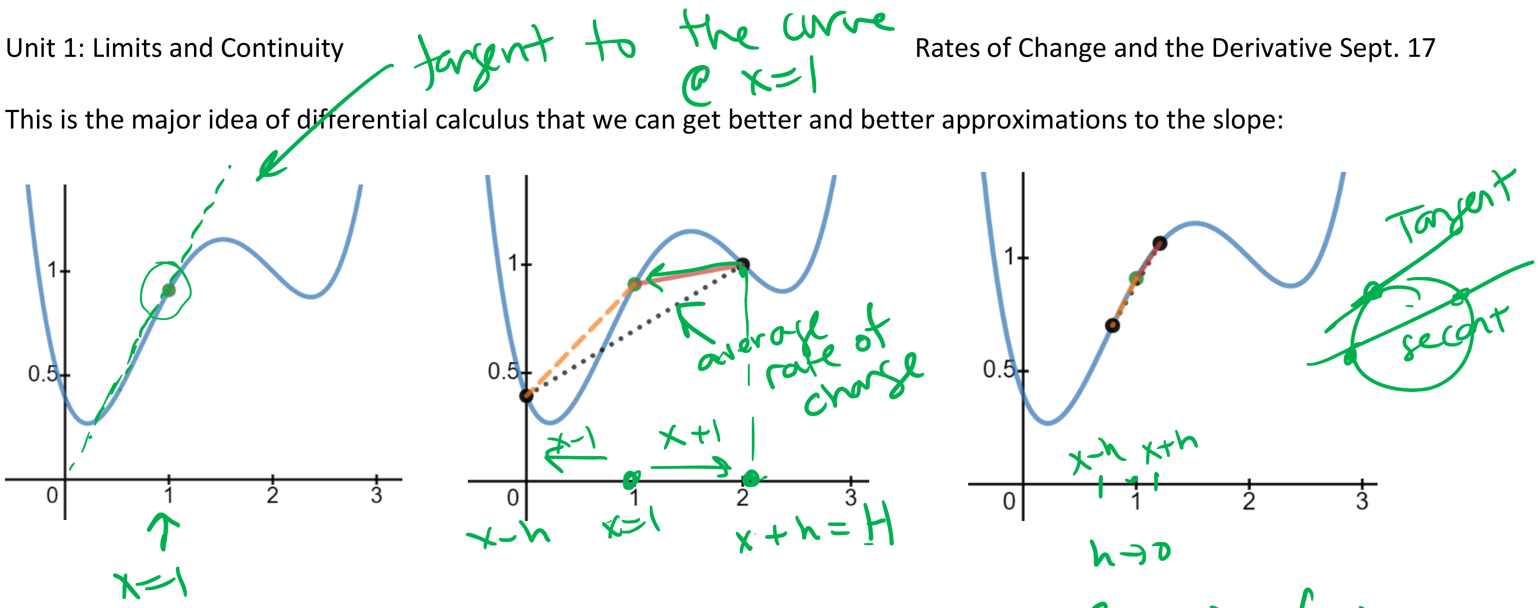
$$\frac{\Delta y}{\Delta x} = \frac{218-85}{2} = 66.5$$

small

$$\frac{\Delta y}{\Delta x} = \frac{313-4}{8} = 39$$

If the number of Twiitter users on year $t$ can be modelled by $T(t)$, how could we get a better approximation for the rate of growth in 2012?

This is the major idea of differential calculus that we can get better and better approximations to the slope:




dotted $\frac{f(x)-f(x-h)}{h}$ slope $\frac{f(x+h)-f(x)}{h}$ solid

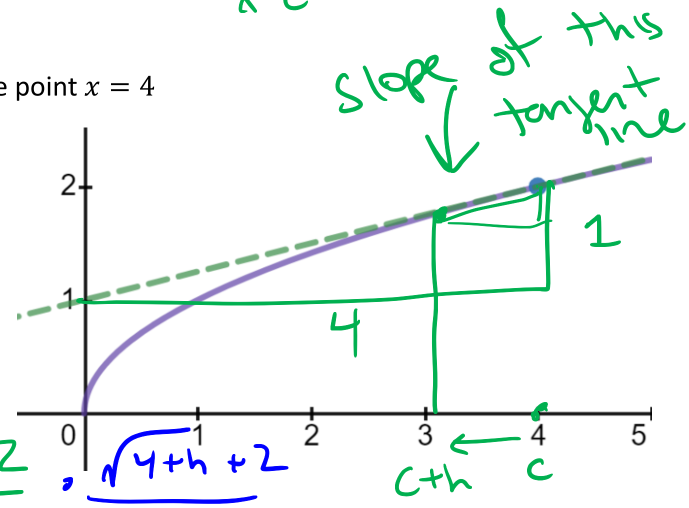
**Definition**: The slope of a function $f$ at the point $x = c$ is the value of the following limits

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{H \to c} \frac{f(H)-f(c)}{H-c} = \text{slope @ } x=c$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

Consider using this definition to find the slope of $f(x) = \sqrt{x}$ at the point $x = 4$

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(c)-f(c+h)}{-h} = \frac{f(c+h)-f(c)}{h}$$




$$m = \lim_{h \to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$= \lim_{h \to 0} \frac{4+\cancel{h}-4}{\cancel{h}(\sqrt{4+h}+2)} = \frac{1}{4}$$

point 1 point 2

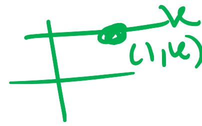
$$\lim_{x \to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \frac{1}{4}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(c)}{x-c} = \frac{f(c+h)-f(c)}{h}$$



**Practice**: Determine the slope of the functions at the indicated point and find the equation to the tangent line. Determine the slope using both limit definitions.

a. $f(x) = k$, $@\, x = 1$ $(1,k)$




$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{k-k}{h} = 0$$

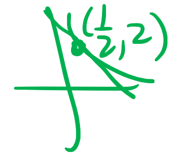
Tangent line $y = \underbrace{0\cdot(x-1)+k}_{\text{point/slope}} = k$

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{k-k}{x-1} = 0$$

$$y - y_0 = m(x-x_0)$$

$$\boxed{y = m(x-x_0)+y_0}$$

b. $g(x) = \frac{1}{x}$, $@\, x = \frac{1}{2}$




$f(x) = \sqrt[4]{x}$ $@\, x=c$

$f(x) = x^3$ $@\, x=c$

c. $h(x) = mx + b$, $@\, x = c$

d. $k(x) = ax^2$, $@\, x = c$

When we compute the slope at an arbitrary point $x = c$ we are basically finding the slope at any point of $x$. We have a special name for the slope at any point $x$ and that is the **derivative**.

**Notation**: There are two major ways we denote the derivative of a function $f(x)$ and the slope at $x = c$

**Newton**:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \underbrace{f'(x)}_{\text{f prime}}$$

; ' (prime) is an operator

$$f \xrightarrow{\text{deriv.}} \underbrace{f'(x)}_{\text{slope of } f}$$

**Liebniz**:

dee by dee x

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{d}{dx}\left[f(x)\right] = \frac{df}{dx} = \frac{dy}{dx} \neq \frac{\Delta y}{\Delta x}$$

derivative of f with respect to x

not a fraction! operator

**WARNING!**

$f'(2) \Rightarrow$ slope @ $x=2$

$\frac{d}{dx}\underbrace{f(2)}_{\text{const}} = 0$

$\frac{d}{dx}f(x)\Big|_{x=2} \to$ slope @ $x=2$

Once we're comfortable with the notation, we can use it as operators in computers and describe objects as the derivative.

For example, consider $f(x) = \frac{\cos x^2}{x^2+1}$, what is the slope at $x = 1$?

$f'(1) = -1.111\ldots$

Desmos

$$\frac{d}{dx}\left(\frac{\cos x^2}{x^2+1}\right)\Bigg|_{x=1}$$

$$-\sin(1) - \frac{\cos 1}{2}$$

**Practice**: Given that $f$ is continuous and $f'(1) = 2$ and that $f(1) = 3$, how could we approximate $f(1.05)$?

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|---|
| **Practice Problems**: 2.4 #1-6 (select), 7, 8, 19-22<br>3.1 #2-10, 25 |
| **Textbook Readings**: Page 82-85, 95-100 |
| **Workbook Practice**: Page 88-90 |
| **Next Day**: Differentiability |