Rates of Change and Derivative

Goal:
- Can approximate rate of change from data
- Can determine the precise rate of change of a function using limits.
- Understands the derivative as the slope of the tangent line
- Can use Newton and Leibnitz notation for the derivative
- Understands the difference between $\frac{d}{dx}f(a)$ and $f'(a)$
- Can use calculator to determine the derivative at a point

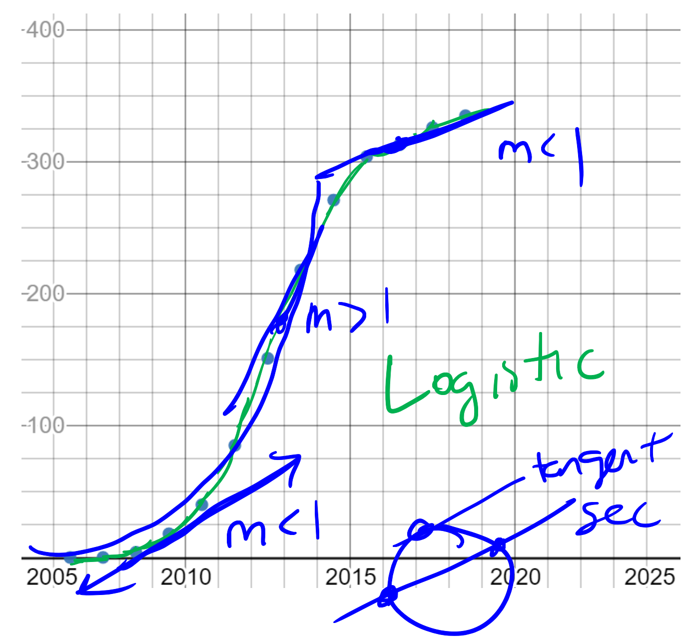
Terminology:
- Average rate of change
- Tangent
- Derivative
- $\frac{d}{dx}$ and $f'(x)$

t	2006	2008	2010	2011	2013	2014	2016	2018
$T(t)$	0.02	4	40	85	218	271	313	335

https://www.statista.com/statistics/282087/number-of-monthly-active-twitter-users/

Above are selected values of the time t measured during the middle of the year in June for the number of active Twitter users.

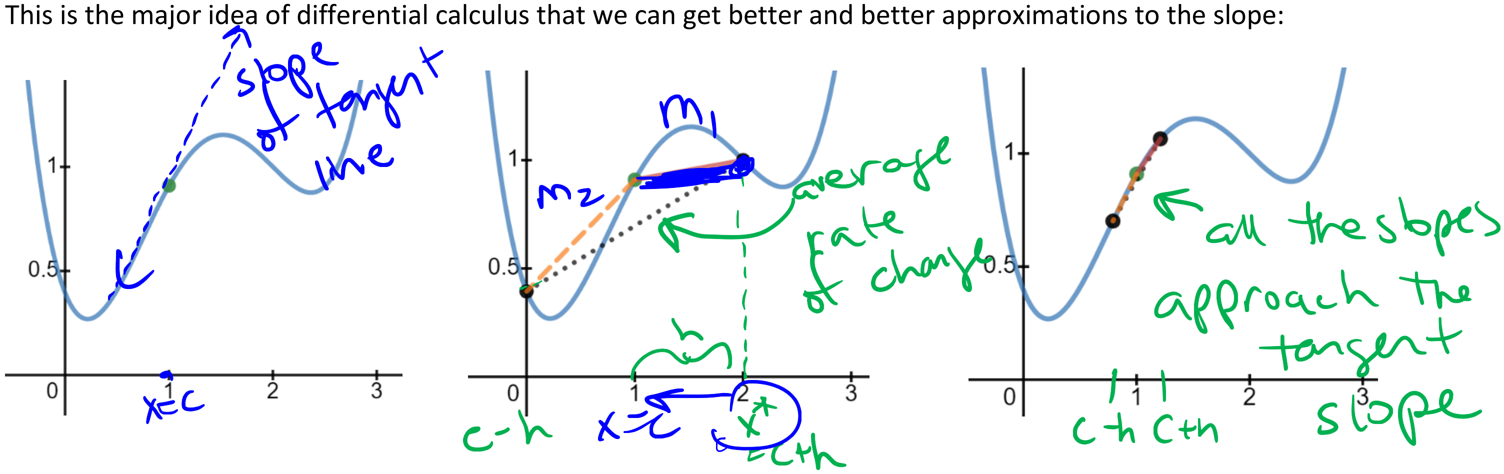
On the board:
How has the rate of Twitter's growth changed over the past 15 years?



Give at least two approximations for the rate Twitter was growing in 2012

If the number of Twiitter users on year t can be modelled by $T(t)$, how could we get a better approximation for the rate of growth in 2012?

This is the major idea of differential calculus that we can get better and better approximations to the slope:



let h→0 and look at $\frac{\Delta y}{\Delta x}$ → tangent slope

Definition: The slope of a function f at the point $x = c$ is the value of the following limits

$\frac{\Delta y}{\Delta x} \to$ limit

$$\boxed{\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

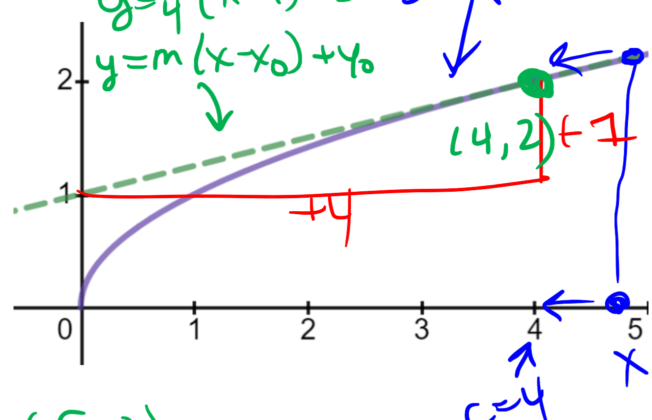
Consider using this definition to find the slope of $f(x) = \sqrt{x}$ at the point $x = 4$

$$\frac{\Delta y}{\Delta x}$$

$$\lim_{x \to 4} \frac{\overbrace{f(x) - f(4)}^{\Delta y}}{\underbrace{x - 4}_{\Delta x}} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

$$= \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \frac{1}{4}$$



$$\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h \to 0} \frac{\cancel{4}+h-\cancel{4}}{\cancel{h}(\sqrt{4+h}+2)} = \frac{1}{4}$$

Practice: Determine the slope of the functions at the indicated point and find the equation to the tangent line. Determine the slope using both limit definitions.

a. $f(x) = k$, $@\, x = 1$

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{k-k}{h} \quad \frac{0}{0}?$$

This $= 0$

h NOT 0

$$= 0$$

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{k-k}{x-1} = 0$$

$$y = 0(x-1)+k$$

$$y = k$$

b. $g(x) = \frac{1}{x}$, $@\, x = \frac{1}{2}$

$$\lim_{h\to 0}\frac{\frac{1}{\frac{1}{2}+h}-2}{h} = \lim_{h\to 0}\frac{\frac{2}{1+2h}-2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{2}-\cancel{2}-4\cancel{h}}{\cancel{h}(1+2h)} = -4$$

$$\lim_{x\to 1/2}\frac{\frac{1}{x}-2}{x-\frac{1}{2}} = \lim_{x\to 1/2}\frac{2\overset{-1}{\cancel{(1-2x)}}}{x\cancel{(2x-1)}}$$

$$= -4$$

$$y = -4\left(x-\tfrac{1}{2}\right)+2$$

c. $h(x) = mx + b$, $@\, x = c$

$$\lim_{x\to c}\frac{\cancel{mx}+\cancel{b}-\cancel{mc}-\cancel{b}}{\cancel{x-c}} = m$$

$$\lim_{h\to 0}\frac{m(\cancel{c}+\cancel{h})+\cancel{b}-\cancel{mc}-\cancel{b}}{\cancel{h}} = m$$

$$y = m(x-c)+mc+b$$

$$= mx+b$$

d. $k(x) = ax^2$, $@\, x = c$

$$\lim_{x\to c}\frac{ax^2-ac^2}{x-c} = \lim_{x\to c}\frac{a\cancel{(x-c)}(x+c)}{\cancel{x-c}}$$

$$= 2ac$$

$$\lim_{h\to 0}\frac{a(c+h)^2-ac^2}{h} = 2ac$$

$$\frac{\Delta y}{\Delta x}$$

When we compute the slope at an arbitrary point $x = c$ we are basically finding the slope at any point of x. We have a special name for the slope at any point x and that is the **derivative**. $\Rightarrow$ slope @ x

Notation: There are two major ways we denote the derivative of a function $f(x)$ and the slope at $x = c$

1669 **Newton**: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ f prime at x

prime is an operator

1673 **Liebniz**: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{d}{dx}[f(x)] = \frac{df}{dx}$ dee f by dee x

$\equiv$ derivative of f with respect to x

measure of how much f changes as x changes

WARNING!

$f'(1) \equiv$ slope @ $x=1$ $\frac{d}{dx}f(1) = 0$ $\frac{d}{dx}(f(x))\Big|_{x=1}$

constant

Once we're comfortable with the notation, we can use it as operators in computers and describe objects as the derivative.

For example, consider $f(x) = \frac{\cos x^2}{x^2+1}$, what is the slope at $x = 1$?

$f'(1) = -1.11...$ $\frac{d}{dx}\left(\frac{\cos x^2}{x^2+1}\right)\Big|_{x=1} = -\sin 1 - \frac{\cos 1}{2}$

Desmos wolfram alpha

Practice: Given that f is continuous and $f'(1) = 2$ and that $f(1) = 3$, how could we approximate $f(1.05)$?

Practice Problems: 2.4 #1-6 (select), 7, 8, 19-22 3.1 #2-10, 25	
Textbook Readings: Page 82-85, 95-100	
Workbook Practice: Page 88-90	
Next Day: Differentiability	