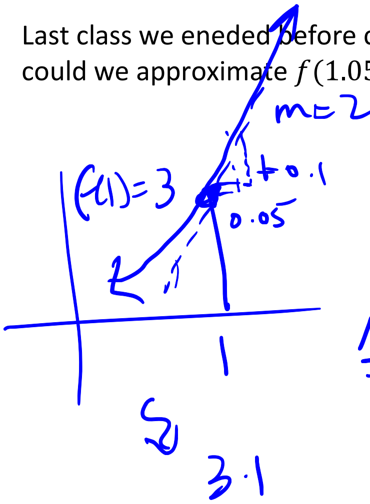
Differentiability and Continuity

Goal:
- Can determine if a function is differentiable based on its graph and using the limit definition.
- Understands why differentiability implies continuity.

Terminology:
- Local linearity
- Differentiable

$\sin x \approx x$ @ $x=0$

Last class we eneded before consider this problem: Given that f is continuous and $f'(1) = 2$ and that $f(1) = 3$, how could we approximate $f(1.05)$?



$\frac{\Delta y}{\Delta x} = 2$

1.05 close to 1

$\Rightarrow \underbrace{f(1.05)}_{\text{limit}}$ close to $\underbrace{f(1)}_{\text{value}}$

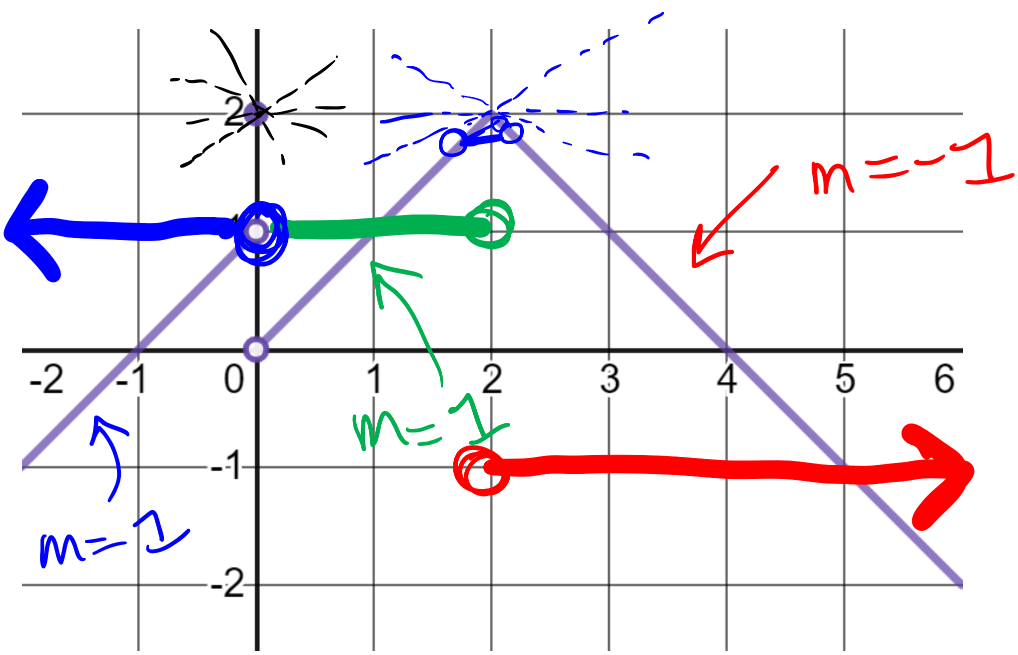
b/c continuous

→ use tangent line

$y = 2(x-1)+3$ $x = 1.05$
$= 3.1$

The idea is that if we don't move far away, then the tangent line looks like the graph: That is the graph is locally linear if it has a derivative.

Consider the following graph of f, we want to find f'



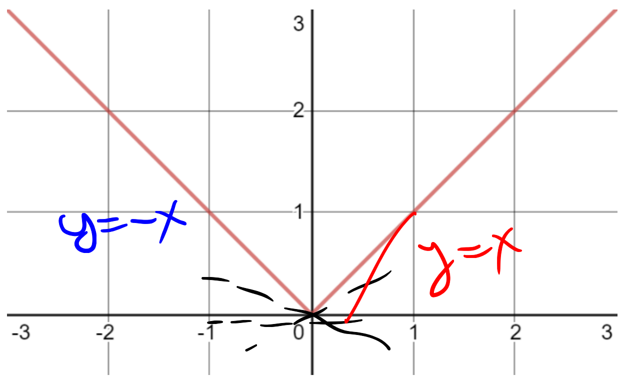
$f'(0)$, $f'(2)$ is undefined

We say a function is **differentiable** at $x = c$ if $f'(c)$ exists. This means the limits below exist

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

This leads us to a few conditions that would make the function NOT differentiable at a point.

1. Corner, such as $f(x) = |x|$ at $x = 0$



Goal show $f'(0)$ has a problem

Show $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ $\boxed{\text{when } c=0}$

has a problem

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

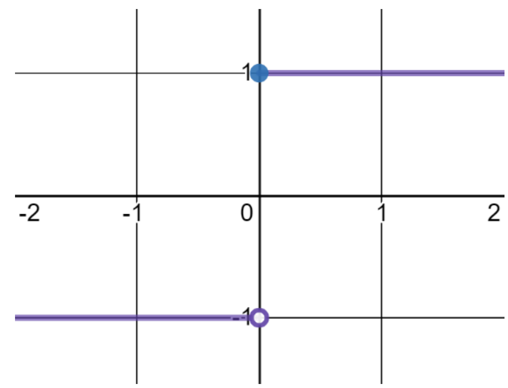
$$\lim_{x\to 0}\frac{f(x)-0}{x} = \lim_{x\to 0}\frac{f(x)}{x}$$

$$\lim_{x\to 0^+}\frac{x}{x} = 1$$

$$\lim_{x\to 0^-}\frac{-x}{x} = -1$$

NOT The same

2. Jump, such as $g(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ at $x = 0$



$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}$$

$$\lim_{h\to 0}\frac{g(h)-1}{h}$$

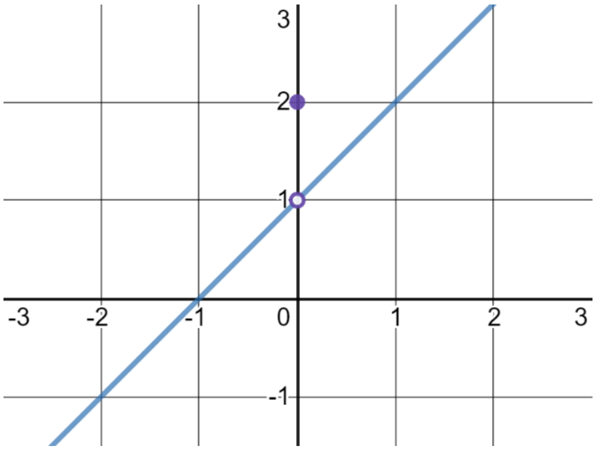
$$\lim_{h\to 0^+}\frac{1-1}{h} = 0$$

$$\lim_{h\to 0^-}\frac{-1-1}{h} = +\infty$$

NOT The same

$f'(0)$ undefined

3. Hole, such as
$$h(x) = \begin{cases} x+1, & x \neq 0 \\ 2, & x = 0 \end{cases} \quad \text{at } x = 0$$



$$\lim_{x \to 0} \left(\frac{h(x) - h(0)}{x - 0} \right)$$

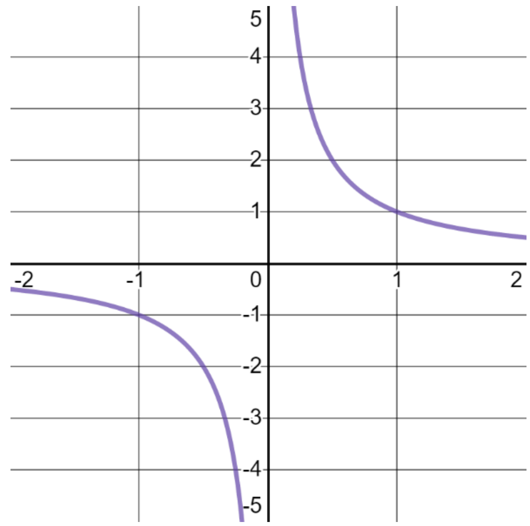
$$= \lim_{x \to 0} \frac{(x+1) - 2}{x}$$

$$= \lim_{x \to 0} \frac{x-1}{x}$$

$$\lim_{x \to 0^+} \frac{x-1}{x} = -\infty$$

$$\lim_{x \to 0^-} \frac{x-1}{x} = +\infty$$

4. Asymptote, such as $k(x) = \frac{1}{x}$ at $x = 0$

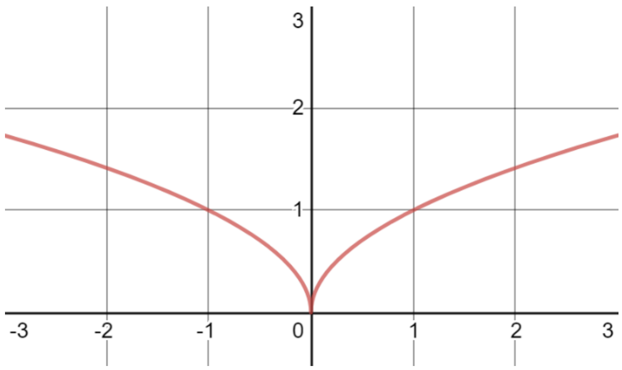


$$\lim_{x \to 0} \left(\frac{k(x) - k(0)}{x - 0} \right)$$

But $k(0)$ is undefined so this limit makes no sense.

$k'(0)$ is undefined.

5. Cusp, such as $p(x) = \sqrt{|x|}$ at $x = 0$



$$\lim_{x\to 0} \frac{p(x)-p(0)}{x-0}$$

$$=\lim_{x\to 0} \frac{\sqrt{|x|}-0}{x}$$

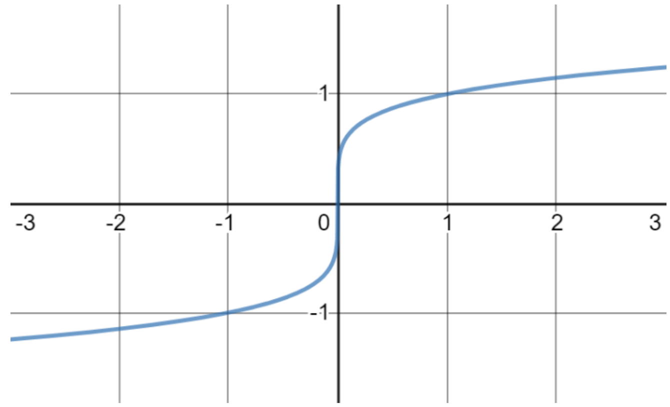
$$=\lim_{x\to 0^+} \frac{\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$$

$$\lim_{x\to 0^-} \frac{\sqrt{-x}}{x} = \lim_{x\to 0^+} \frac{\sqrt{x}}{-x} = -$$

$$=\lim_{x\to 0^+} \frac{1}{-\sqrt{x}} = -\infty$$

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{|x|}}{x} \text{ DNE}$$

6. Vertical Tangent, such as $q(x) = \sqrt[5]{x}$ at $x = 0$



$$\lim_{x\to 0} \frac{q(x)-q(0)}{x-0}$$

$$=\lim_{x\to 0} \frac{\sqrt[5]{x}-0}{x}$$

$$=\lim_{x\to 0} \frac{1}{x^{4/5}} = \infty$$

So the slope is ∞ which means the slope (ie $q'(0)$) is undefined.

A criticial consequence of these cases is that: If a function is differentiable at $x = c$ is MUST be continuous at $x = c$.

Proof: If f is differentiable $\Rightarrow$ f is continuous

If f is NOT continuous $\Rightarrow$ f is NOT dif.

assume f is differentiable $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \in \mathbb{R} = f'(c)$

Show that f is continuous $\boxed{\lim_{x \to c} f(x) = f(c)}$

$= \lim_{x \to c} f(x) - f(c) = 0$

$$\lim_{x \to c} \left(f(x) - f(c) \right) \cdot \frac{x-c}{x-c} = \boxed{\lim_{x \to c} \frac{f(x) - f(c)}{x - c}} \cdot (x - c)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} x - c$$

$$= f'(c) \cdot 0 = 0$$

☐ QED

Differentiable is smooth

see mrlcgullen.weebly.com

Practice Problems:	2.4 #1-6 (select), 7, 8, 19-22 3.1 #2-10, 25
Textbook Readings:	Page 3.2
Workbook Practice:	Page 91-99
Next Day:	Online learning derivative rules (chapter 3 of the textbook and workbook)