Log Laws

KNOW	**DO**	**UNDERSTAND**
The basic log laws and the change of base law	Can use the log laws to simplify expressions and evaluate logs of different bases.	*Function Characteristics*: Can determine the domain of a sum of logarithms *Transformations:* Can relate horizontal transformations to vertical transformations using log laws
Vocab & Notation • Change of base		

In grade 9 and 10 you learned about the exponent laws and know that

$$b^x \cdot b^y = b^{x+y} \qquad (b^x)^y = b^{xy} \qquad b^{-1} = \frac{1}{b}$$

$$\underbrace{b \cdot b \cdots b}_{x} \cdot \underbrace{b \cdot b \cdots b}_{y} = \underbrace{b \cdot b \cdots b}_{x+y} \qquad \underbrace{b^x \cdot b^x \cdots b^x}_{y} = b^{\overbrace{x+\cdots+x}^{y}} \qquad \text{By definition}$$

Using function notation if $g(x) = b^x$ then the above laws for exponents give unique and defining characteristics:

$$\underbrace{g(n) \cdot g(m)}_{\text{product outside}} = g(\underbrace{n+m}_{\text{sum inside}}) \qquad \underbrace{g(n)^m}_{\text{outside exponent}} = g(\underbrace{nm}_{\text{inside product}})$$

Logarithms, being the inverse of exponentials, have similar laws:

Product Law: $\log_b(m \cdot n) = \log_b m + \log_b n$ $f(x) = \ln x$

$f(m \cdot n) = f(m) + f(n)$

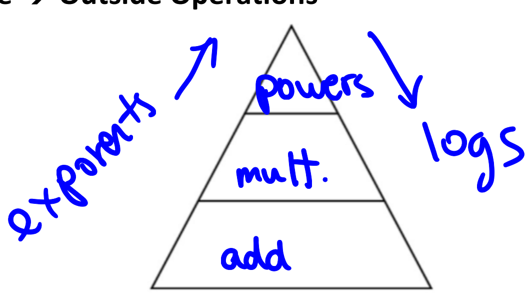
INSIDE product   OUTSIDE sum

Power Law: $\log_b(x^n) = n \cdot \log_b x$

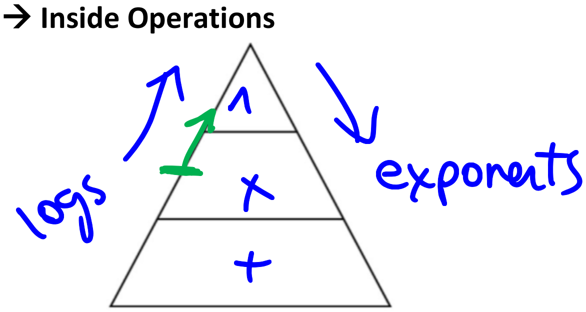
Inside power   outside product $f(x^n) = n \cdot f(x)$

Quotient Law: $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ $f\left(\frac{x}{y}\right) = f(x) - f(y)$

Inside → Outside Operations



Outside → Inside Operations



$A = B + C$

Product Law Proof: want to show $\ln xy = \ln x + \ln y$

say $\ln xy = A$ $\quad \ln x = B$ $\quad \ln y = C$

$xy = e^A$ $\quad x = e^B$ $\quad y = e^C$

$\Rightarrow e^A = e^B \cdot e^C = e^{B+C} \Rightarrow A = B + C$

QED proof done $\square$

Power Law Proof: $\ln x^n = n \ln x$

$$\ln x^n = \ln(\underbrace{x \cdot x \cdots x}_{n \text{ times}}) = \underbrace{\ln x + \ln x + \cdots + \ln x}_{n \text{ times}}$$

$$= n \cdot \ln x$$

So just like we would simplify exponential functions we can simplify logs.

$$e^x \cdot \left(\frac{e^y}{e^z}\right)^2$$

$e^x \cdot (e^{y-z})^2$

$e^{x+2y-2z}$

$$\ln x + 2(\ln y - \ln z)$$

$\ln x + 2 \cdot \ln \frac{y}{z}$

$\ln x + \ln\left(\frac{y}{z}\right)^2$

$\ln\left(x \cdot \frac{y^2}{z^2}\right)$

Practice: Use log laws to simplify the following into a single log:

$$\log 7 - \log 3 + \log 6$$

$\log 7 - (\log 3 - \log 6)$

$\log 7 - \log(3/6)$

$\log\left(\frac{7}{3/6}\right) = \log 14$

$$3\ln 6 - \ln 9 - \ln 8$$

$\ln 6^3 - \ln 9 - \ln 8$

$\ln\left(\frac{6^3}{9 \cdot 8}\right)$

$= \ln(3)$

$$-\frac{1}{2}\ln 81 - 2\ln 3$$

$\ln 81^{-1/2} - \ln 3^2$

$\ln \frac{1}{9} - \ln 9$

$\ln\left(\frac{1}{81}\right)$

$$-3\log 2 + (2\log 7 - \log 5)$$

$\underline{-\log 2^3} + \log 7^2 \underline{-\log 5}$

$\log\left(\frac{7^2}{2^3 \cdot 5}\right)$

$$2\log_2(12+3) - (\log_2 5 + \log_2 4)$$

$\log_2 15^2 - \log_2 20$

$\log_2\left(\frac{15^2}{20}\right)$

$= \log_2\left(\frac{75}{4}\right)$

$\ln 10 - \ln 5$ * $\quad \frac{\ln 10}{\ln 5} \neq \ln\frac{10}{5}$

$\ln(10 \cdot 5)$ $\quad \frac{1}{\ln 5} \cdot \ln 10$

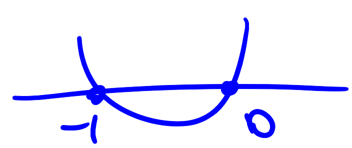
$\left(\ln 10^{\frac{1}{\ln 5}}\right)$

We need to be careful about the domain when we simplify log functions:

Example: Simplify the following and state the overall domain.

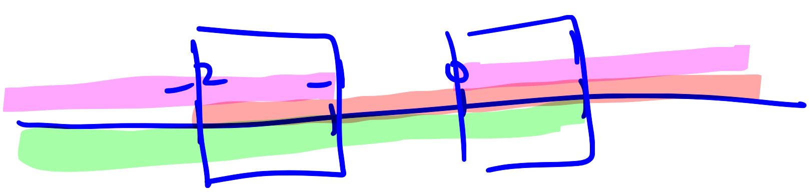
$$f(x) = -\ln(x+2) + 2\ln(1-x) - \ln(x(x+1))$$

$$f(x) = \ln\left(\frac{(1-x)^2}{(x+2)\,x(x+1)}\right)$$



$x+2>0$ $\quad 1-x>0$ $\quad x(x+1)>0$

$x>-2$ $\quad x<1$ $\quad x>0$ or $x<-1$



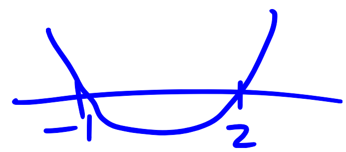
$\Rightarrow$ domain

$x \in (-2,-1) \cup (0,1)$

Practice: Simplify the following and state the overall domain

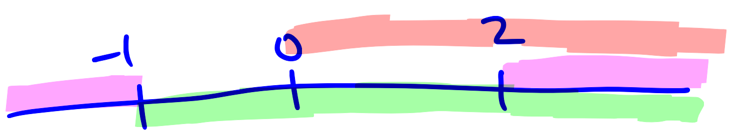
$$g(x) = \log x + 2\log(x+1) - \log\big((x+1)(x-2)\big)$$

$$g(x) = \log\left(\frac{x(x+1)}{(x-2)}\right)$$



$x > 0$ $\quad$ $x+1 > 0$ $\quad$ $(x+1)(x-2) > 0$

$x > -1$ $\quad$ $x < -1$ or $x > 2$



$\boxed{\text{Domain } x > 2}$

Change of Base Law: $\boxed{\log_b a} = \frac{\log_x a}{\log_x b} = \frac{\ln a}{\ln b} = \frac{\log a}{\log b}$

Proof:

$\log_b a = Y \Rightarrow a = b^Y \Rightarrow \ln a = \ln b^Y$

$\Rightarrow \ln a = Y \ln b$

$\Rightarrow \frac{\ln a}{\ln b} = Y = \log_b a$ $\square$

Practice: Evaluate the following logarithms

$\log_2 20 = \frac{\ln 20}{\ln 2} = 4.32$

$\frac{\log 20}{\log 2} = 4.32\ldots$

$\log_5 1000$

$\frac{\log 1000}{\log 5} = \frac{3}{\log 5} = 4.29\ldots$

$\log_\pi e$

$\frac{\ln e}{\ln \pi} = \frac{1}{\ln \pi} = 0.87\ldots$

$\log_{\sqrt{2}} \sqrt{8}$

$\frac{\ln 8^{1/2}}{\ln 2^{1/2}} = \frac{\ln 8}{\ln 2} = \frac{\ln 2 \cdot 3}{\ln 2} = 3$

Practice Problems: 8.3 page 400-403 # 1-3, 5-12, 20, C1, C2