SOLUTIONS: Mixing Practice Problems

1. A tank has a current chlorine concentration of 0.1 g/L and has a volume of 500 L. A more concentrated solution of chlorine (0.5 g/L) is added at rate of 5 L/min. Consider the two situations:
   a. Water leaves the tank at the same rate of 5 L/min
   b. Water is being drained from the tank slower at a rate of 2 L/min.

   Write a differential equation for each situation and solve the differential equation in part A. Determine the amount of chlorine in the tank after 15 minutes for each case.

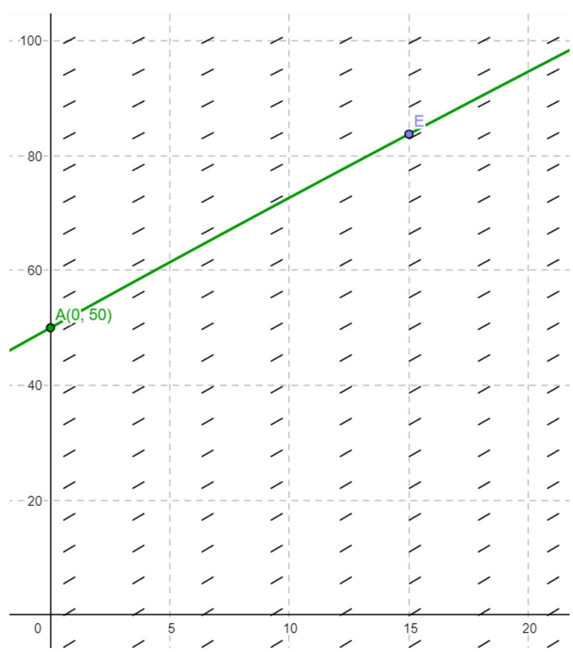
   a. Let c be the amount of chlorine in grams, then the (stable) steady state will be $500 \cdot 0.5 = 250$ g
   $$\frac{dc}{dt} = k(c - 250)$$
   Where $k = \frac{-5}{500} = -0.01$

   Alternatively, we can use rate in and rate out
   $$\frac{dc}{dt} = 2.5 - \frac{5c}{500}$$
   Both are the same and we get $\Gamma = c - 250 \Rightarrow \Gamma = Ce^{-0.01t} = c - 250$ and the initial mass of chlorine is $500 \cdot 0.1 = 50$ g
   $$c = -200e^{-0.01t} + 250$$
   $$\Rightarrow c(15) = 77.9 \text{ g}$$

   b. This time we must use rate in and rate out
   $$\frac{dc}{dt} = 2.5 - \frac{2c}{V(t)}$$
   Where the volume at time t is $V = 500 + 3t$
   $$\frac{dc}{dt} = 2.5 - \frac{c}{250 + 1.5t}$$

   If we graph it and zoom in, I predict the mass of chlorine to be about 84 g. Similar to the above, but slightly more makes sense.

   For those interested the solution to the differential equation is
   $$c(t) = C\left(t + \frac{500}{3}\right)^{-\frac{2}{3}} + \frac{3}{2}\left(t + \frac{500}{3}\right)$$



2. A population of insects would double every two months without any outside factors. If birds are eating the insects at a rate of 1000 insects/bird per day, then consider the following. (Assume 30-day months)
   a. The population of birds is constant at 100.
   b. The population of birds grows due to migration at a rate of 10 new birds per month (after starting at 100)

   Write a differential equation for each situation and solve part A. Determine the number of insects after 1 year if there were 10 million to start for each case.

   a. If the population would double every two months we know that the insect population on its own would be modeled by

$$\frac{dP}{dt} = kP \Rightarrow P = P_o e^{kt}$$

And $P(2) = 2P_0 = P_0 e^{2k} \Rightarrow k = \frac{1}{2}\ln 2 = 0.3466 \text{ month}^{-1}$

Now we use population in and population out. Note that 100 bird would eat 3 million insect each month. Let P be the population in million of insect.

$$\frac{dP}{dt} = 0.3466P - 3$$

Let $\Gamma = P - 8.6555$ and then $\frac{dP}{dt} = 0.3466(\Gamma + 8.6555) - 3 = 0.3466\Gamma$

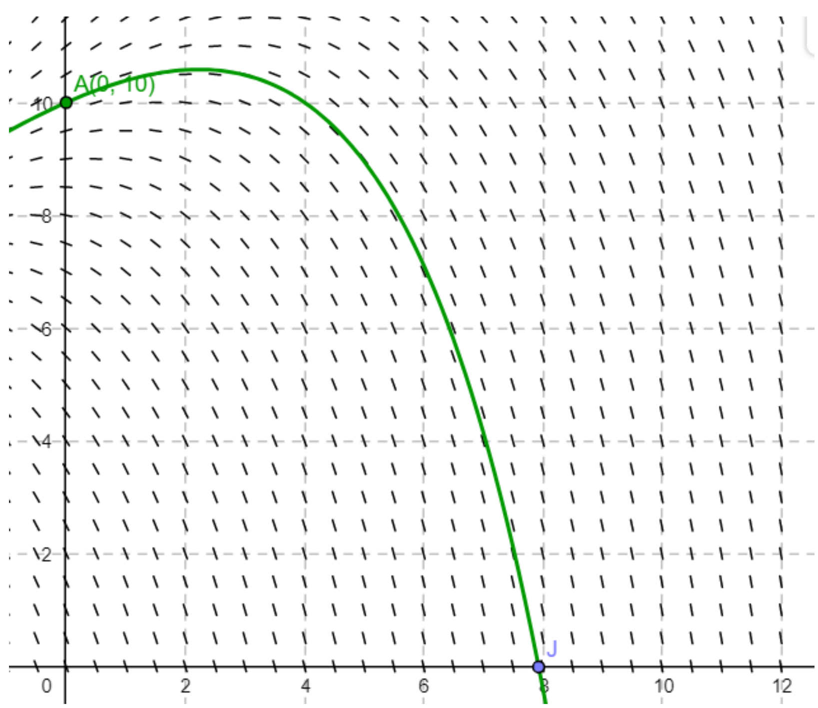
$$\Rightarrow \Gamma = Ce^{0.3466t} = P - 8.6555, \qquad P(0) = 10$$

$$P = 1.3445e^{0.3466t} + 8.6555$$
$$P(12) = 94.731 \text{ million}$$

   b. Now the birds eat $1000 \cdot (100 + 10t) \cdot 30$ insect each month. Divide by 1 million and you get

$$\frac{1\cancel{000}(10\cancel{0} + 1\cancel{0}t)3\cancel{0}}{1\,\cancel{000\,000}} = \frac{3(10+t)}{10}$$
$$\frac{dP}{dt} = 0.3466P - (3 + 0.3t)$$

Once we graph it we see that the insects have all died after 8 months ☹



For those interested the solution to the differential equation is

$$P(t) = Ce^{kt} + \frac{3}{k} + \frac{0.3}{k^2} + \frac{0.3t}{k}$$

Where $k = \frac{1}{2}\ln 2$ and $C = 10 - \frac{3}{k} - \frac{0.3}{k^2}$

3. If you plan to retire you want to know how much you need to save so that you can live off what you've saved and invested. Say you plan to retire at age 55 and expect to live for another 30 years being relatively active. You want to live off a comfortable \$100 000 a year and expect safe investments to return an annual interest rate of 4%.
   a. What is the steady state investment amount?
   b. Determine the minimal you need to have by age 55 so that your savings will not be depleted until the 31st year of retirement.
   c. At retirement you find you have saved 1 million dollars. How much can you afford to pay yourself for 30 years?

a. For the differential equation with A the amount in millions

$$\frac{dA}{dt} = 0.04A - 0.1$$

The steady state is of course $0.04 = 0.1 \Rightarrow A = 2.5$ million dollars

b. The minimum we need is finding A_0 such that $A(0) = A_0$ and $A(31) = 0$
Solving the differential equation by letting $\Gamma = A - 2.5$ we get $\Gamma = Ce^{0.04t} = A - 2.5$

$$A_0 = Ce^0 + 2.5$$
$$0 = Ce^{1.24} + 2.5$$

The second equation tells us that $C = -2.5e^{-1.24} = -0.723$ and so we can solve for A_0 and get

$$A_0 = 1.777$$

So we need about 1.8 million dollars to retire at 55 if we pay ourselves 100K a year.

c. If instead we now have a differential equation

$$\frac{dA}{dt} = 0.04A - p$$

Where p is your annual pension. Let $\Gamma = A - \frac{p}{0.04}$ so we have

$$\frac{dA}{dt} = 0.04\left(\Gamma + \frac{p}{0.04}\right) - p = 0.04\Gamma \Rightarrow \Gamma = Ce^{0.04t} = A - \frac{p}{0.04}$$

And we have $A(0) = 1$ and $A(31) = 0$

$$1 = Ce^0 + \frac{p}{0.04}$$
$$0 = Ce^{1.24} + \frac{p}{0.04}$$

Subtract the two to solve for C

$$1 = C - Ce^{1.24} \Rightarrow C = \frac{1}{1 - e^{1.24}} = -0.407$$

And we can solve for p

$$1 = -0.407 + \frac{p}{0.04}$$
$$p = 0.056$$

So, we can afford to pay ourselves \$56 000 per year for 30 years.