Factoring Polynomial

KNOW	**DO**	**UNDERSTAND**
How to find the remainder using remainder theorem. How to test a factor using factor theorem	Factor polynomials Use long division to divide polynomials	*Function Characteristics*: That the output of a polynomial at any point $x = c$ is the remainder after being divided by $x - c$. Polynomials in $\mathbb{Z}[X]$ can be prime and have similar properties to integers (gcd/lcm, etc).

Vocab & Notation

- Set of polynomials with integer coefficients: $\mathbb{Z}[X]$
- Irreducible polynomial
- Factor Theorem
- Integer Root Theorem & Rational Root Theorem
- Remainder Theorem

$12 = 24 \times 0.5$

We are motivated to factor polynomials just like we factor integers, but to do so we need to be able to divide polynomials. As we are motivated to factor like integers, we are only going to look at polynomials whose coefficients are integers.

Definition: Let $\mathbb{Z}[X]$ be the set of polynomials with integer coefficients. From now on, any polynomial I think about will be in this set and we will factor in this set.

$2x^2 - 1 \in \mathbb{Z}[X]$ $\quad \frac{1}{2}x^3 \notin \mathbb{Z}[X]$

For example, factor 4199. Primes

2 go into 4199?

3?

$4+1+9+9 = 23$

5?

7?

11? $4-1+9-9 = 3$

$$\begin{array}{r} 599 \\ 7\overline{)4199.0} \\ -3500.0 \\ \hline 699.0 \\ -630.0 \\ \hline 69.0 \\ -63.0 \\ \hline 6.0 \end{array}$$

remainder

$$\begin{array}{r} 323 \\ 13\overline{)4199} \\ -3900 \\ \hline 299 \\ -260 \\ \hline 39 \\ -39 \\ \hline 0 \end{array}$$

remainder

$\Downarrow$

$4199 = 13 \cdot 323$

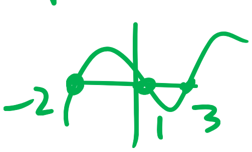
We want to take these ideas and help us factor:

$$p(x) = x^4 - x^3 - 7x^2 + 13x - 6$$

$p(x) = (x-a) \cdot q(x)$

$= (x-a) \cdot (x^3 + \dots + c) = x^4 + \dots \underbrace{-a \cdot c}_{-6}$

$\star$ $p(a) = 0$



$(x+2), (x-1), (x-3)$

Theorem: The **factor theorem** states that if $(x-a)$ divides $p(x)$ then $p(a)=0$.

constant term

But how do we find a?

Theorem: The **integer root theorem** states that if $(x-a)$ is a factor of $p(x)=x^n+\cdots+C$ then a is a factor of C.

*Note: there is a larger version of this theorem called the **rational root theorem**: given the polynomial $p(x)=Ax^n+\cdots+C$, if $x=\frac{p}{q}$ is a zero we will have that p is a factor of C and q is a factor of A.

So, let's factor this polynomial!

$$p(x)=x^4-x^3-7x^2+13x-6$$

what are the factors?

$\pm1, \pm2, \pm3, \pm6$

does $p(a)=0$?

$(x-1)$; $(x+1)$; $(x-2)$; $(x+2)$; ...

possible factors.

Test $p(1)=1-1-7+13-6=0$!

$\Rightarrow (x-1)$ is a factor

$p(x)\div(x-1) \Rightarrow 4199\div13$

$$\begin{array}{r|l} & x^3 \quad -7x \quad +6 \\ x-1 & x^4-x^3-7x^2+13x-6 \\ & -(x^4-x^3) \\ \hline & \qquad -7x^2+13x-6 \\ & \qquad -(-7x^2+7x) \\ \hline & \qquad\qquad 6x-6 \\ & \qquad\qquad -(6x-6) \\ \hline & \qquad\qquad 0 \end{array}$$

$$\Rightarrow \begin{array}{r|l} & 3 \quad 2 \quad 3 \\ 13 & 4\times10^3+1\times10^2+9\times10+9 \\ & -(3\times10^3 \quad 9\times10^2 \quad 0 \quad 0) \\ \hline & \quad 2 \quad 9 \quad 9 \\ & \quad 2 \quad 6 \quad 0 \\ \hline & \qquad 3 \quad 9 \\ & \qquad 3\,9 \\ \hline & \qquad 0 \end{array}$$

$x^3-7x+6 = p_1(x)$.

Try $p_1(1)=1-7+6=0$

$$\begin{array}{r|l} & (x^2+x-6) \\ x-1 & x^3-7x+6 \\ & -(x^3-x^2) \\ \hline & x^2-7x+6 \\ & -(x^2-x) \\ \hline \end{array}$$

$$\begin{array}{r} -6x+6 \\ -6x+6 \\ \hline 0 \end{array}$$

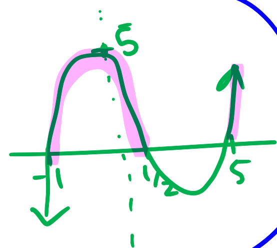
$$p(x)=(x-1)^2(x-2)(x+3)$$

Practice: Factor the polynomials and sketch them. State the intervals the polynomial is positive.

$$2x^3 - 9x^2 - 6x + 5 = P_1(x)$$

$P_1(-1) = -2 - 9 + 6 + 5 = 0$

$$\begin{array}{r} 2x^2 - 11x + 5 \\ x+1 \overline{)\, 2x^3 - 9x^2 - 6x + 5} \\ -(2x^3 + 2x^2) \\ \hline -11x^2 - 6x + 5 \\ -(-11x^2 - 11x) \\ \hline 5x + 5 \end{array}$$



$\Rightarrow P_1(x) = (x+1)(2x^2 - 11x + 5)$

$= (x+1)(2x-1)(x-5)$

$P_1(x) > 0$ when $-1 < x < \frac{1}{2}$ or $x > 5$

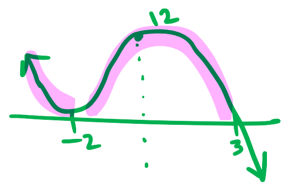
$$-x^3 - x^2 + 8x + 12 = P_2(x)$$

$P_2(-2) = 8 - 4 - 16 + 12 = 0$

$$\begin{array}{r} -x^2 + x + 6 \\ x+2 \overline{)\, -x^3 - x^2 + 8x + 12} \\ -(-x^3 - 2x^2) \\ \hline x^2 + 8x + 12 \\ -(x^2 + 2x) \\ \hline 6x + 12 \end{array}$$

$\Rightarrow P_2(x) = (x+2)(-x^2 + x - 6)$

$= -(x+2)(x-3)(x+2)$



$P_2(x) > 0$ when $x < 3,\ x \neq -2$

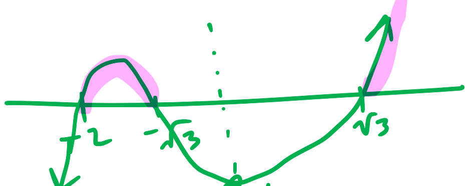
$$x^3 + 2x^2 - 3x - 6$$

$(x+2)(x^2-3)$

$x^2 - 3 = 0$ when $x = \pm\sqrt{3}$ not rational

$x^2 - 3$ is prime (irreducible) in $\mathbb{Z}[x]$

in $\mathbb{R}[x]$ then yes!



$P_3(x) > 0$ when $-2 < x < -\sqrt{3}$ or $x > \sqrt{3}$

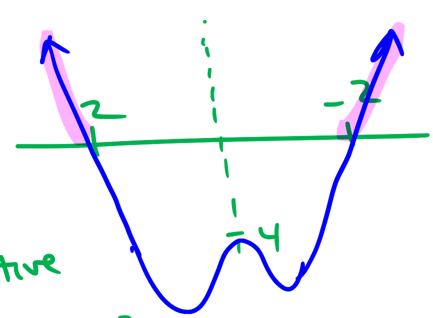
$$x^4 - 3x^2 - 4$$

$(x-2)(x+2)(x^2+1)$

$x^2 + 1 = 0$

$x^2 = -1$

$x^2 + 1$ is prime in $\mathbb{R}[x]$



Positive when $|x| > 2$

We can also divide polynomials by polynomials that are not factors and end up with a remainder.

Example: $435 \div 7$

$$\begin{array}{r} 62 \\ 7\overline{)435} \\ \underline{42} \\ 15 \end{array} \qquad \begin{array}{r} 15 \\ \underline{14} \\ 1 \end{array} \text{ remainder}$$

$$\frac{435}{7} = 62 + \frac{1}{7}$$

$$\frac{p(x)}{(x-a)} = q(x) + \frac{r}{x-a}$$

Example: Simplify the following quotient:

$$\frac{x^4 - 2x^2 + 5x + 1}{x + 2}$$

$$\begin{array}{r} x^3 - 2x^2 + 2x + 1 \\ x+2\overline{)x^4 - 2x^2 + 5x + 1} \\ -(x^4 + 2x^3) \\ \hline -2x^3 - 2x^2 + 5x + 1 \\ -(-2x^3 - 4x^2) \\ \hline 2x^2 + 5x + 1 \\ -(2x^2 + 4x) \\ \hline x + 1 \end{array} \qquad \begin{array}{r} x+1 \\ -(x+2) \\ \hline -1 \end{array} \text{ remainder}$$

$$x^3 - 2x^2 + 2x + 1 - \frac{1}{x+2}$$

$p(-2) = -1$!

Theorem: The **remainder theorem** says if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$

$$\frac{p(x)}{x-a} = q(x) + \frac{r}{x-a} \Rightarrow p(x) = q(x)\,(\cancel{x-a})^{0} + r$$

$$p(a) = r$$

Practice: Determine the remainder of the following quotient using long division and verify using remainder theorem.

$$\frac{-x^3 + 2x^2 + 5x - 2}{x - 3} = p(x)$$

remainder $= p(3) = -27 + 18 + 15 - 2 = 4$

$$\begin{array}{r} -x^2 - x + 2 \\ x-3\overline{)-x^3 + 2x^2 + 5x - 2} \\ -(-x^3 + 3x^2) \\ \hline -x^2 + 5x - 2 \\ -(-x^2 + 3x) \\ \hline 2x - 2 \end{array} \qquad \begin{array}{r} 2x - 2 \\ -(2x - 6) \\ \hline 4 \end{array} \text{ is remainder}$$

Practice Problems: 3.2 page 124 – 125 # 1-3, 5-11, 14, 15, 17, C1, C2   3.3 page 133 – 135 # 1-8, 15, 16, C1