Modelling and Solving Rational Functions

KNOW	DO	UNDERSTAND
What rates are additive and what rates are not.	Can solve a given rational equation. Can estimate a reasonable solution given the context.	*Function Characteristics*: Can model a situation with a rational function and use the model to make predictions that are easy to use and interpret.

Vocab & Notation

- Net Change

In calculus a large portion is working with rates of change and we can start investigating rates with rational functions since rational functions naturally lend themselves to the form of a rate:

Speed: $\frac{\text{distance}}{\text{time}}$

Task Efficiency: $\frac{\text{amount completed}}{\text{time}}$

Density: $\frac{\text{mass}}{\text{volume}}$

Concentration: $\frac{\text{mass disolved}}{\text{volume}}$ or $\frac{\text{volume disolved}}{\text{volume}}$

We need to be careful when we think of how these things combine when we try to add two rates.

Example: A boat can travel 40 km/h relative to the water and the river moves at a speed of 8 km/h. What is the speed of the boat upstream and downstream?

upstream

40 km/h − 8 km/h = 32 km/h

down

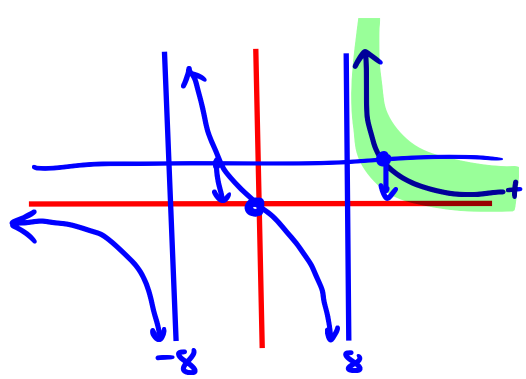
40 km/h + 8 km/h = 48 km/h

The boat's speed is b, make an equation for the total time it takes to travel 30 km upstream and 30 km downstream.

$\Rightarrow$ speed $= \frac{\text{dist}}{\text{time}}$ $\Rightarrow$ time $= \frac{\text{dist}}{\text{speed}}$.

$$T(b) = \frac{30 \text{ km}}{b-8 \text{ km/h}} + \frac{30 \text{ km}}{b+8 \text{ km/h}}$$

$$= \frac{60b}{(b-8)(b+8)}, \quad b > 8$$



$b >> 0$

$\Rightarrow T > 0$

Determine the speed of the boat if the round trip takes 1.5 hours.

$$T(b) = 1.5 = \frac{60b}{b^2-64}$$

$$1.5b^2 - 96 = 60b$$

$$1.5\underline{b^2 - 60b} - 96 = 0$$

$$1.5(b^2 - 40b) - 96 = 0$$

$$\downarrow$$

$$1.5(\underbrace{b - 20}) ^2 - 96 - 600 = 0$$

$$(\underline{b^2 - 40b} + 400)$$

$$1.5(b-20)^2 = 696$$

$$(b-20)^2 = 464$$

$$b - 20 = 21.54$$

$$b = 41.54 \text{ km/h}$$

Practice Determine the speed of the boat if a round trip takes 2 hours.

$$2 = \frac{60b}{b^2-64}$$

$$\underline{2b^2 - 60b} - 128 = 0$$

$$2(b^2 - 30b) - 128 = 0$$

$$2(b - 15)^2 - 128 - 450 = 0$$

$$\downarrow$$

$$\underline{b^2 - 30b} + 225$$

$$\Rightarrow 2(b-15)^2 = 578$$

$$b - 15 = 17$$

$$b = 32 \text{ km/h}$$

Example: At a grocery store, the express line can have 15 people move through every 10 minutes, and the regular line they have 12 people move through every 45 minutes. How efficient are they together?

Express

$\frac{15 \text{ ppl}}{10 \text{ min}}$

Regular

$\frac{12 \text{ ppl}}{45 \text{ min}} = \frac{2.67 \text{ ppl}}{10 \text{ min}}$

Together

$\frac{17.67 \text{ ppl}}{10 \text{ min}} = \text{Express} + \text{Regular}$

If there are x people going through the line every 10 minutes in the express side, determine an equation for the time it would take to get N customers through.

$\text{Rate} = \frac{\text{ppl}}{10 \text{ min}} \Rightarrow \text{min} = \frac{\text{ppl}}{10 \cdot \text{Rate}}$

$$T(x, N) = \frac{N}{10\left(\frac{2.67}{10} + \frac{x}{10}\right)}$$

It takes the express lane cashier typically 60 seconds to complete one purchase, whereas it takes the regular cashier typically 500 seconds to complete one purchase. If both were working together, how long would it take them to complete one purchase?

Express

60 sec/1 purch

1 pur/60 sec

$\frac{1}{60}$ pur/sec

Regul

500 sec/1 purch

1 pur/500 sec

$\frac{1}{500}$ pur/sec

Together

~~560 sec/1 purch~~

$\Rightarrow$ 0.0187 pur/sec

= 1 pur/53.57 sec

If it takes the regular lane 850 seconds to complete one purchase and working together the regular and express lane take 100 seconds, how fast does it take to go through the express lane alone?

Reg

1 pur/850 sec

Together

1 pur/100 sec

Express

Estimate 1 purch/200 sec

$\frac{1}{100}$ pur/sec $= \frac{1}{850}$ pur/sec $+ \frac{1}{t}$ pur/sec

$\frac{1}{t} = \frac{1}{100} - \frac{1}{850} \Rightarrow t = 113.3 \text{ sec}$

Practice: Two hoses are being used to fill up a kiddie pool to get the job done faster. The hose attached to the outside of the house can fill the pool up on its own in t_o hours. The hose attached to the kitchen in the house would take 5 hours to fill up the pool on its own. Determine t_0 if it takes 1.5 hours for pool to be filled up using both hoses.

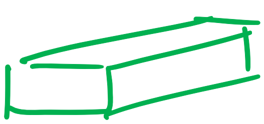
outside	inside	together
1 pool	1 pool	1 pool
t_o hours	5 hrs	1.5 hrs

$$\Rightarrow \frac{1}{t_o} + \frac{1}{5} = \frac{1}{1.5}$$

$$\Rightarrow t_o = 2.16 \text{ hrs}$$

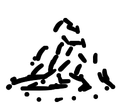
Building Models from a Rational Relationship

Example: Steel is (mostly) made from combining iron and carbon. If iron has a density of 7.9 $\mathrm{g/cm^3}$ and carbon has a density of 2.1 $\mathrm{g/cm^3}$, and we mix 900 g of iron with 100 g of carbon, determine the density of the steel mixture.



Fe $\to \rho = 7.9\ \mathrm{g/cm^3}$, $m = 900\ \mathrm{g}$

ρ = rho



C $\to \rho = 2.1\ \mathrm{g/cm^3}$, $m = 100\ \mathrm{g}$

$V = \frac{100}{2.1}\ \mathrm{cm^3}$

estimate of density

$2.1 \leq \quad 7.7 \quad \leq 7.9$

$\rho_{Total} = 7.9\ \mathrm{g/cm^3} + 2.1\ \mathrm{g/cm^3} = 10\ \mathrm{g/cm^3}$ ← not true!

$$\rho_{Total} = \frac{M_{Total}}{V_{Total}} = \frac{100+900}{\frac{100}{2.1} + \frac{900}{7.9}} = 6.2\ \mathrm{g/cm^3}$$

Practice: Using the previous densities, what would the density be if instead 700 g of iron was used and 300 g of carbon was used.

$$\text{density} = \frac{700+300}{\frac{700}{7.9}+\frac{300}{2.1}}$$

$$= 4.3\ g/cm^3$$

Generalize this. The density of iron is ρ_{Fe} and the density of carbon is ρ_{C}. We add m_{Fe} grams of iron and m_{C} of carbon. Determine the density of the steel mixture.

$$\text{Density} = \frac{m_{Fe}+m_c}{\left(\frac{m_{Fe}}{\rho_{Fe}}+\frac{m_c}{\rho_c}\right)} = \left[\frac{m_{Fe}+m_c}{m_{Fe}\rho_c+m_c\rho_{Fe}}\right]\rho_{Fe}\cdot\rho_c$$

$$P(m_c, m_{Fe}) = \frac{(m_{Fe}+m_c)\,\rho_{Fe}\,\rho_c}{m_{Fe}\rho_c+m_c\rho_{Fe}}$$

$$\boxed{P: \mathbb{R}^2 \longrightarrow \mathbb{R}}$$

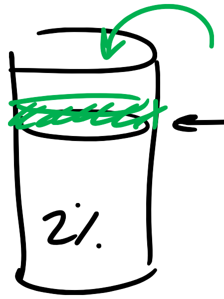
$m_{Fe} = 1000 - m_c$

$$P: [0,\infty)^2 \rightarrow (0,\infty)$$

Example: A 2 L jug of 2% milk (fat % by volume) is mixed with 50 mL of 18% cream (fat % by volume). What is the concentration of the mixed solution?

estimate

$2 < 5\% < 18$



50 mL 18% $\Rightarrow$ 9 mL fat

2L

2000 mL $\Rightarrow$ 40 mL fat

$$\text{Fat}\% = \frac{\text{Fat Vol}}{\text{Tot Vol}} = \frac{40+9}{2000+50} = 2.4\%$$

Practice: If a 4 L jug of 2.5 % milk was mixed with 75 mL of whipping cream (33%). What is the concentration of fat of the mixed solution?

$$\text{Fat}\% = \frac{\text{Fat Vol}}{\text{Tot Vol}} = \frac{4000 \times 2.5\% + 75 \times 33\%}{4000 + 75} = 3.1\%$$

Generalize this. The fat percentage of milk is p_m and the fat percentage of cream is p_c. The volume of milk is V_m and the volume of cream is V_c. Determine the concentration of the mixture.

$$F(V_m, V_c, \rho_c, \rho_m) = \frac{V_m \cdot \rho_m + V_c \cdot \rho_c}{V_m + V_c}$$

$$F: \mathbb{R}^4 \to \mathbb{R}$$

Practice Problems: 9.2 page 452 – 456 # 12, 13
9.3 page 465 – 467 # 1, 3-6, 12-17