

Chapter 1 Review

TRANSFORMATIONS

To transform a function, f , we can stretch it or compress it in the vertical or horizontal directions, reflect it over either primary axis, and shift it in the vertical or horizontal directions.

In function notation these transformations look like:

$$a \cdot f\left(\frac{1}{b}(x - c)\right) + d$$

And the equivalent mapping notation looks like:

$$(x, y) \mapsto (bx + c, ay + d)$$

Where a, b are vertical and horizontal stretch factors respectively. If they are negative, there is a reflection over an axis. The vertical and horizontal shift amounts are given by k and h . Written in this manner expansions/compressions/reflections must be done before translations since we follow order of operations in the mapping notation.

***** THIS DOES NOT MEAN IT IS IMPOSSIBLE TO SHIFT BEFORE WE STRETCH!!** If you see $f(cx + d)$ you can interpret this as a horizontal shift d units and then a stretch by c units OR put it in the standard form $f\left(c\left(x + \frac{d}{c}\right)\right)$ as a horizontal stretch by c units and then a shift by d/c units.

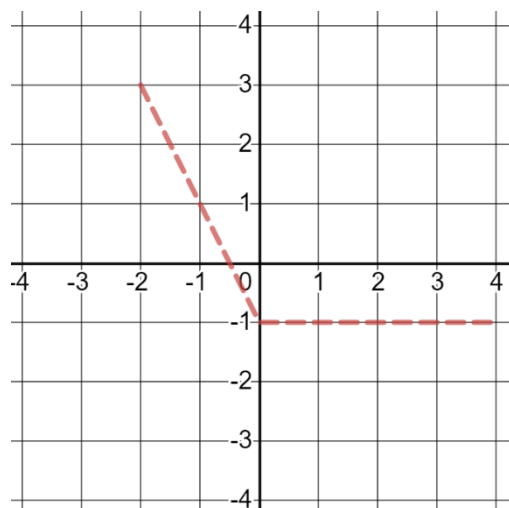
Note that mapping notation does exactly what it says horizontally and vertically. If it multiplies by a number with absolute value larger than 1, then it will be an expansion as the values get larger. If it multiplies it by a small number with absolute value less than 1, then it will be a compression as the value gets smaller. Positive shifts are right and up and negative shifts are left and down.

Big Ideas you need to know and understand:

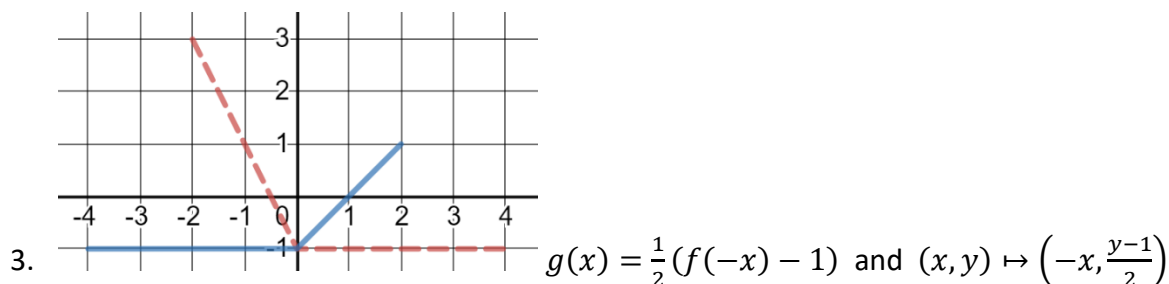
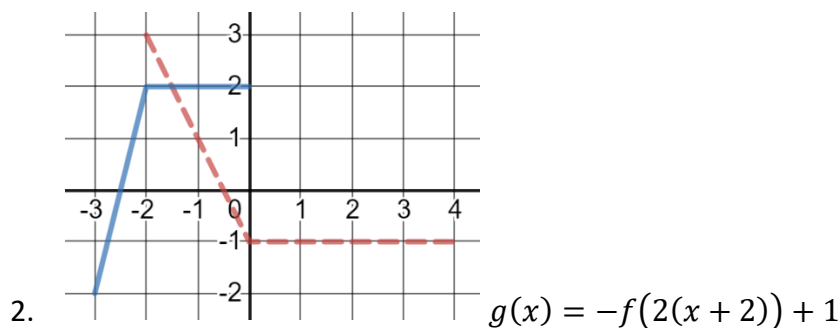
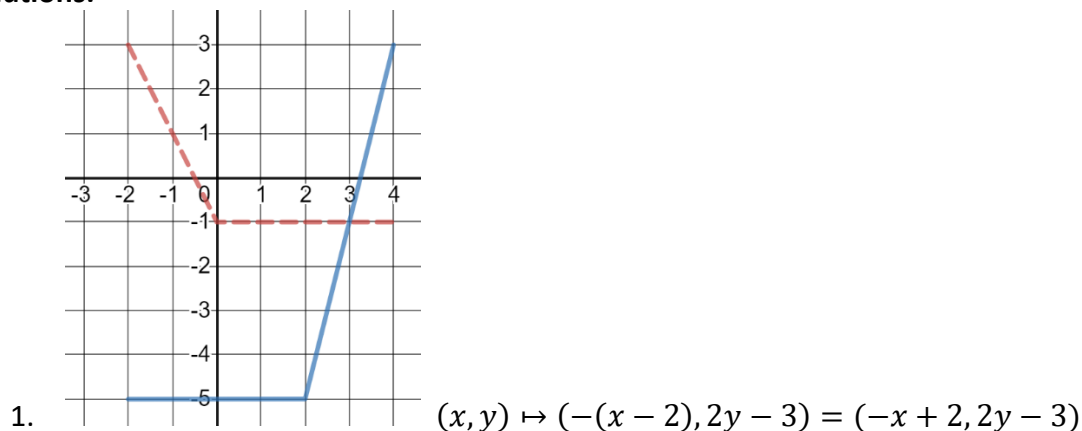
- Mapping notation does transformation in order of operation and nothing is reversed. Function notation reverses the horizontal transformations
- Can take either a set of verbal instruction, function notation, or mapping notation and create the other 2 from it and graph the image.
- To find a graph of the image function, we can either do each transformation step by step, or use mapping notation to see where coordinates get mapped to.

Review questions: Given the graph of $f(x)$ graph the image $g(x)$ and write the mapping notation and function notation of each transformation.

1. $g(x) = 2f(-x + 2) - 3$
2. $(x, y) \rightarrow \left(\frac{1}{2}x - 2, -y + 1\right)$
3. Shift it down by 1, then compress vertically by 2. Reflect it over the y -axis



Solutions:

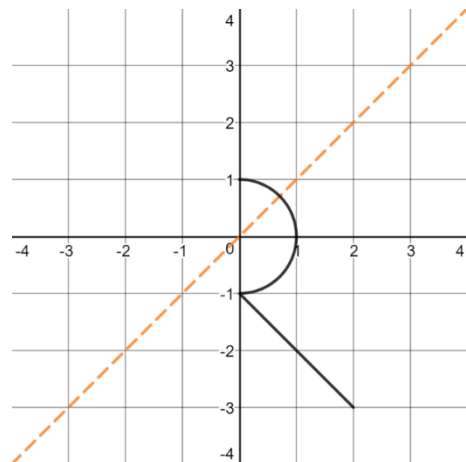
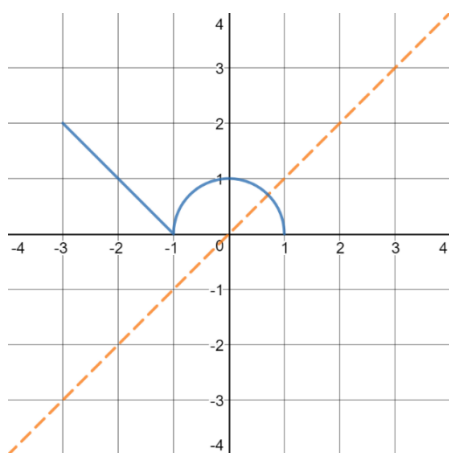


INVERSES OF FUNCTIONS

When we invert a function, we go in the reverse order, that is from the range to the domain. If we have an ordered pair (x, y) it is implied that $x \mapsto y$ so the inverse would be $y \mapsto x$ and is the coordinate (y, x) . To graph an inverse, simply graph the new (y, x) coordinates. This amounts to reflecting the graph over the line $y = x$ since all points in the form (a, a) will stay the same and be invariant.

Reflect over the line $y = x$. Notice some key points that get mapped are:

- $(-3, 2) \mapsto (2, -3)$
- $(-1, 0) \mapsto (0, -1)$
- $(0, 1) \mapsto (1, 0)$
- $(1, 0) \mapsto (0, 1)$



Review

Some functions, like the one above, are not 1-to-1 (some outputs can be mapped to from different inputs and fails the horizontal line test) and as a result their inverses are not functions (the same input will yield different outputs and it fails the vertical line test). To describe such an inverse as a function, the domain of the original function needs to be restricted so it is 1-to-1.

Big Ideas you need to know and understand:

- All 1-to-1 functions have an inverse function. If not 1-to-1 just adjust the domain so that the function is 1-to-1 (passes the horizontal line test).
- Since the original function is just a series of instructions in order, the inverse is the same instructions in the reverse order and opposite operation so that everything is undone.

Review questions:

Find the inverse of the following functions

4. $f(x) = \sqrt{x+5} - 3$
5. $f(x) = \frac{x+4}{x+1}$
6. $f(x) = x^2 - 4$ that includes the point $x = 1$
7. $f(x) = -2(x-3)^2$ that includes the point $x = 1$

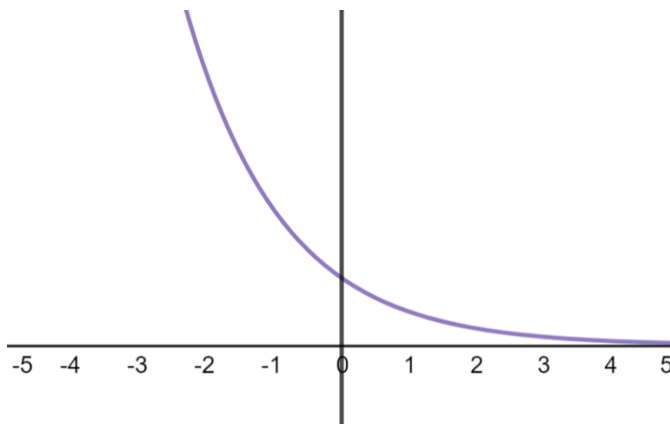
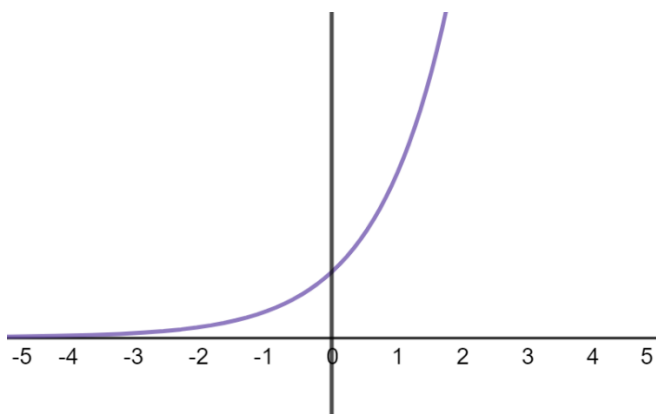
Solutions:

4. $f^{-1}(x) = (x+3)^2 - 5$
5. $f^{-1}(x) = \frac{x-4}{1-x}$
6. $f^{-1}(x) = \sqrt{x+4}$, we need that $x \geq 0$ in $f(x)$
7. $f^{-1}(x) = -\sqrt{-\frac{x}{2}} + 3$, we need that $x \leq 3$ in $f(x)$

Chapter 7 and 8 Review

GRAPHING THE EXPONENT

The basic exponential function is $f(x) = b^x$ for some base $b > 0$ and $b \neq 1$. We don't allow the base to be negative because then the function f is undefined infinitely often (taking even roots of a negative number is not allowed), and if the base is 0 or 1 then the function is a straight line (if $b = 0$ we also have the problem that 0^0 is not well defined).



Review

The two basic shapes (first if $b > 1$, second if $b \in (0,1)$) of the exponential. They both have a horizontal asymptote at $y = 0$ and a y -intercept of 1 (because $b^0 = 1$ for any base) and they will pass through the points $(1, b)$ and $(-1, \frac{1}{b})$.

When graphing a **transformation** of the exponential, say $g(x) = af(b(x - h)) + k$ where $f(x) = B^x$, understand that the transformations change the characteristics of the function in different ways. That is:

- The asymptote is only affected by vertical shifts (ie the asymptote is d)
- The point $(0,1)$ vertical stretches and shifts (and horizontal shifts if any). It will move to $(c, a + d)$ so you can identify the shift left right and the stretch since you already know d . Remember this point is usually a distance 1 from the asymptote and the vertically stretch changes this distance.
- The point $(1, b)$ is affected by everything but it is usually 1 unit to the left of $(0,1)$ so horizontal stretches will change this distance.

Big Ideas you need to know and understand:

You need to know the three basic characteristics of the exponential so you can effortlessly graph the basic exponential.

When graphing a transformed function, understand why the asymptote is only affected by vertical shifts and find the image of $(0,1)$ and $(1, b)$

Review questions: Graph the following functions g based off a transformation of f .

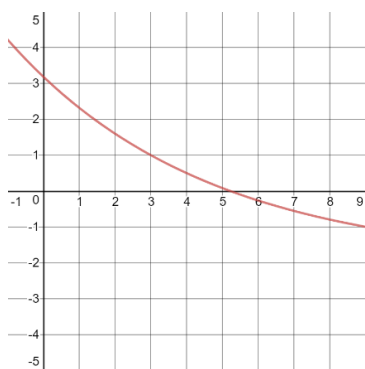
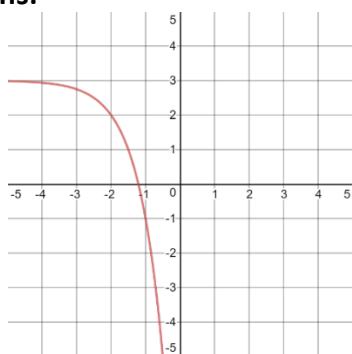
8. $f(x) = 2^x; g(x) = -f(2x + 4) + 3$

9. $f(x) = 1.2^x; g(x) = 3f(3 - x) - 2$

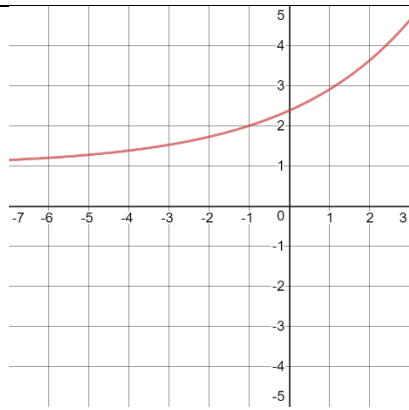
10. $f(x) = 5^x; g(x) = f\left(\frac{x+1}{5}\right) + 1$

11. $f(x) = 0.3^x; g(x) = -f(3x) + 2$

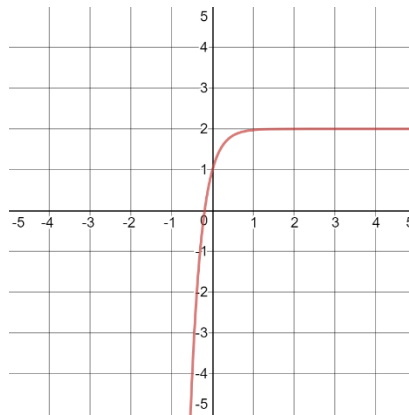
Solutions:



Review



10.



11.

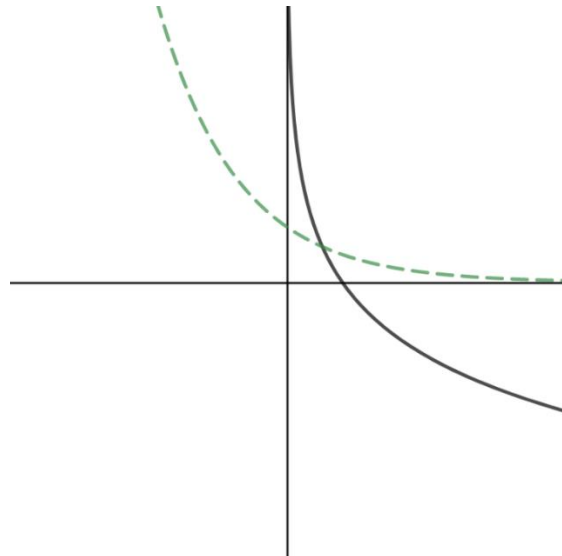
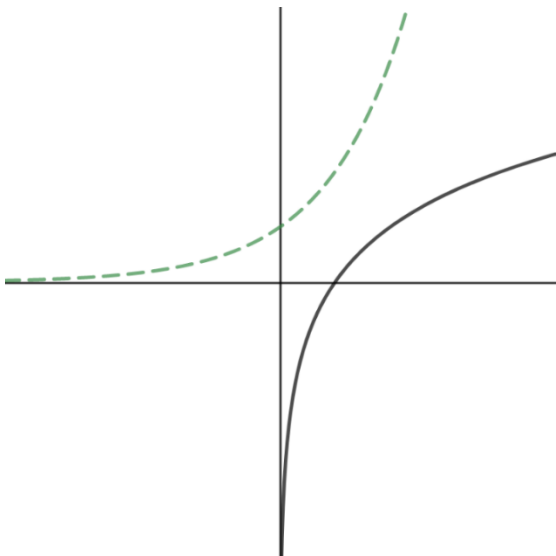
GRAPHING THE LOG

The logarithm is just the inverse of the exponent. Every exponential can be written in an equivalent logarithmic form:

$$a = b^c \Leftrightarrow \log_b a = c$$

By taking the $\log_b()$ of both sides.

From the exponential form it is clear that $b^y = x$ is the inverse of $b^x = y$ so the log is the inverse of the exponential with the same base.



Review

The basic shape of $\log_b x$ is $b > 1$ (on the left) and $b \in (0,1)$ (on the right). Note that the function has a vertical asymptote at $x = 0$ and will pass through the point $(1,0)$ as well as the points $(b, 1)$ and $(\frac{1}{b}, -1)$.

When graphing **transformations** of the log, the idea is the same as with exponential.

- The vertical asymptote is affected by horizontal shifts
- The point $(1,0)$ is affected by horizontal transformations (and vertical shifts if any)
- The point $(B, 1)$ is going to be affected by everything

Just as with the exponent, graph your asymptote first and then build around it.

Big Ideas you need to know and understand:

Understand how to change between logarithmic and exponential form and that they are equivalent forms of each other, not inverses.

When graphing complex log functions, find the vertical asymptote and then change it to exponential form and graph that. Unless the function is log base 10, then just find the asymptote and use the function itself to make a table of values.

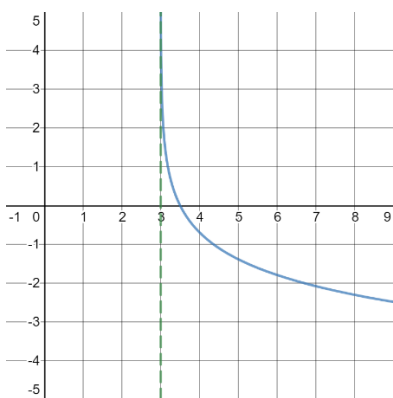
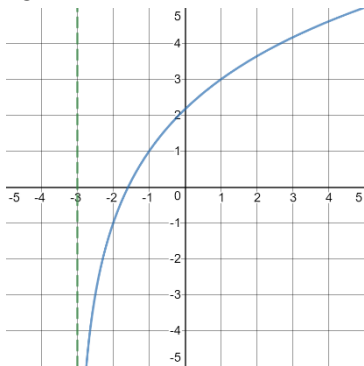
Review questions:

Graph the following

12. $f(x) = \log_2 x$; $g(x) = 2f(x + 3) - 1$

13. $f(x) = \ln x$; $g(x) = -f(2x - 6)$

Solutions:



MODELLING FUNCTION BEHAVIOUR

To model an exponential function, you need to remember that the growth between points is because of multiplication not addition. The function will grow or decay at this value r where every T units of time we multiply the amount we have by r . We then get the total amount, A , at time t is:

$$A(t) = r^{t/T}$$

If we need to adjust the asymptote, then we think about it in terms of transformations. Let d be the maximum or minimum of the model so the equation is then:

$$A(t) = r^{t/T} + d$$

Pick one point on your graph and use that as the image of $(0,1)$. Then we can look at how far away our point is from the asymptote. Typically, it is 1 unit away so if it is a units away that is my vertical stretch:

$$A(t) = a \cdot r^{t/T} + m$$

Finally, our point is typically on the y -axis. We can see how much it was shifted left or right by using the x -coordinate.

$$A(t) = a \cdot r^{(t-c)/T} + m$$

Example: A turkey is cooking in an oven at 325°F (the asymptote). At noon the temperature is 75°F and then 2 hours later the temperature of the turkey is 112.5°F . Make an equation to determine the temperature at 4pm and change your equation to base e to determine when the temperature is 175°F (done).

Solution. We know we pass through $(0, 75)$ and $(2, 112.5)$ using $(\text{time}, \text{temp})$. We also know the asymptote is 325 (degrees Fahrenheit). Our equation must be

$$T(t) = a \cdot r^{t/T} + 325$$

To build the basic part, the first point is 250 units away from the asymptote and the second is 212.5 units away from the asymptote. It takes us 2 hours to change from $250 \rightarrow 212.5$. Our $r = \frac{212.5}{250} = 0.85$ and $T = 2$

$$T(t) = a \cdot (0.85)^{t/2} + 325$$

Finally, the point $(0, 75)$ is still on the y -axis so no shift left or right, but it is 250 units away from the asymptote (the vertical stretch) and it's below the asymptote so there was a reflection over the x -axis $\Rightarrow a = -250$

$$T(t) = -250 \cdot (0.85)^{t/2} + 325$$

So, $T(4) = 144.4^\circ\text{F}$. To change it to base e we need to write $0.85 = e^k$ where $\ln 0.85 = k$

$$\begin{aligned} T(t) &= -250 \cdot (e^k)^{t/2} + 325 \\ &= -250 \cdot e^{\frac{kt}{2}} + 325 \end{aligned}$$

$$\begin{aligned} 175 &= -250 \cdot e^{\frac{kt}{2}} + 325 \\ \frac{175 - 325}{-250} &= e^{kt/2} \\ \ln 0.6 &= \frac{kt}{2} \end{aligned}$$

$$t = \frac{2 \ln 0.6}{k} = \frac{2 \ln 0.6}{\ln 0.85} = 6.3 \approx 6:20\text{pm}$$

Big Ideas you need to know and understand:

Understand how that exponential functions grow by repeated multiplication.

I will tell you the asymptote

Find the length of a growth period by using the two points given to you.

Find the growth/decay rate by looking at heights to the asymptote.

Find the vertical stretch and horizontal shift by picking one point and seeing how it would have moved from (0,1)

Review Questions:

See other hand out.

Log Laws

We discussed the use of 4 major log laws:

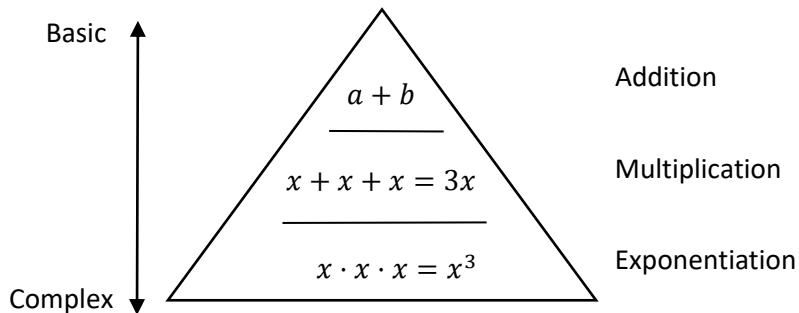
Product: $\log_b(M \cdot N) = \log_b M + \log_b N$

Power: $\log_b x^n = n \cdot \log_b x$

Quotient: $\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$

Change of base: $\log_b a = \frac{\log_x a}{\log_x b}$

The big idea is that logarithms make operations less complex as illustrated below:



The power law takes exponents, as in $\log x^3$, and turns it into a multiplication statement, $3 \cdot \log x$. The product/quotient law takes multiplication/division like $\log 3x$, and turns it into addition or subtraction, $\log 3 + \log x$.

The log of an operation is converted into the operation that is directly above it in simplicity.

These laws are used to simplify and expand logarithms depending on the circumstance. Most often, we will want to simplify logarithms and bring it together under a single logarithm (bring coefficients up as a power and bring sums of logs together as a product). Less often, we will break a single logarithm apart into pieces because we can evaluate the smaller parts (this is where we bring powers down and change a log of a product into a sum of logs).

The change of base is used primarily to evaluate logarithms in terms of the common log or natural log, that is

$$\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

However, we will often use change of base to get a common base among a set of logarithms so that we can implement product or quotient law.

Things we need to know and understand:

- The proof of power law and quotient law
- When simplifying we need to use power law first to get rid of coefficients, then we can use product/quotient laws to bring a sum/difference of logs together as a single log.
- When expanding we need to first expand as a sum/difference using product/quotient laws and then bring down any exponents using power law.

Review Questions:

14. Why does $\log(M + N) \neq \log M + \log N$ and $\log M \cdot \log N \neq \log M + \log N$?
15. Simplify the following as a single logarithm: $\frac{2}{3}\log_5 x - 4\log_{25} y - 3\log_5 z$
16. If $\log_7 2 = a$, $\log_7 3 = b$, and $\log_7 5 = c$, then determine $\log_7 \left(\frac{\sqrt{40}}{63}\right)$ in terms of a, b, c .
17. Write $g(x) = \log_2 x^3 - 3$ as a common log of x (some transformation of $f(x) = \log x$)

Solutions:

14. For the first part the operation "log" does not have the distributive property multiplication has. Basically, multiplication is the only operation that *does* have distributive property (not even division has all of it) so don't do it with logs! For the second part if $\log M = A$ and $\log N = B$ ask yourself does $A \cdot B = A + B$ for every single $A, B \in \mathbb{R}$? No, of course not multiplication is not addition. **DO NOT MAKE THESE COMMON MISTAKES!**

15. $\log_5 \left(\frac{x^{\frac{2}{3}}}{y^2 z^3} \right)$

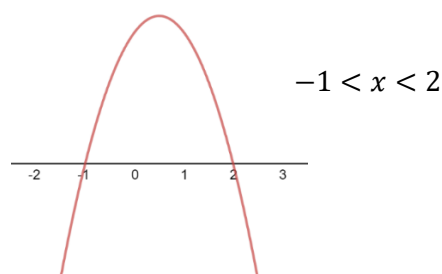
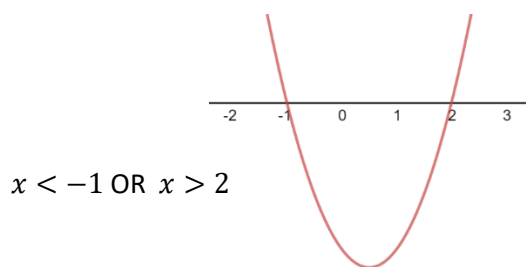
16. $1.5a - 2b + 0.5c - 1$

17. $g(x) = \frac{3}{\log 2} \cdot \log(x) - 3 \approx 10 \log x - 3$

Solving Exponents and Logs Algebraically

When we combine logarithms together, we need to realize that in the simplification process things will cancel or multiply together which can change the sign. Since we cannot take the log of a negative, this will alter the domain. For example, in $2 \log x$ we have that $x > 0$, but this is technically the same function as $\log x^2$ and here $x \neq 0$. The domains don't match so we need to adjust them, so they are both $x > 0$.

When you have a set of logs, each of them must be satisfied and have a positive argument. When considering a parabola to be positive there are two cases. One the left the parabola opens up, and so the values of x will be bound below OR bound above. One the right the parabola opens down, so the values of x will be bound below AND above.



To restrict the domain for a series, write them all out and see what is common with everything.

Review

Solving an equation that involves logarithms is just a process of simplifying it to a single logarithm, change to exponential, and then solve the resulting equation as you have done many times before. Alternatively, you can solve both sides of an equation to a single logarithm and then only consider the input of the log.

Example: Solve

$$2\log(x + 1) - \log x = \log(x - 2)$$

Solution: Note the domain is $x > 2$. We can first move everything to one side and then use log laws

$$\begin{aligned} 2\log(x + 1) - \log x - \log(x - 2) &= 0 \\ \log\left(\frac{(x + 1)^2}{x(x - 2)}\right) &= 0 \end{aligned}$$

Changing to exponential form we get

$$\frac{(x + 1)^2}{x(x - 2)} = 10^0 = 1$$

So that

$$\begin{aligned} (x + 1)^2 &= x(x - 2) \\ x^2 + 2x + 1 &= x^2 - 2x \\ x &= -\frac{1}{4} \end{aligned}$$

There is **NO SOLUTION** because $x > 2$.

Remember that the logarithm is the inverse of the exponential so when we need to solve for an exponent you will need to change it into logarithm form and vice versa if you are solving for the argument in a logarithm.

Things we need to know and understand:

- How to determine the domain of a sum of logs.
- How to solve for a variable in an exponent and how to solve for a variable inside a log.

Review Questions:

18. What is the domain of $f(x) = \log(x + 5) + \log((x + 1)(x - 1)) - \log(x(x - 3))$
19. What is the solution to $\log_2(x + 3) = \log_4((x - 2)(x + 3)) + 1$
20. What is the solution to $\log((x - 5)(x - 2)) + \log((x + 3)(x + 1)) = \log((x - 5)(x + 3))$

Solutions:

18. $-5 < x < -1$ OR $x > 3$
19. $x = \frac{11}{3}$ (domain is $x > 2$)
20. $x = \frac{1 \pm \sqrt{13}}{2} \approx -1.3$ and 2.3 (domain is $x < -3$ OR $x > 5$) so **NO SOLUTION**