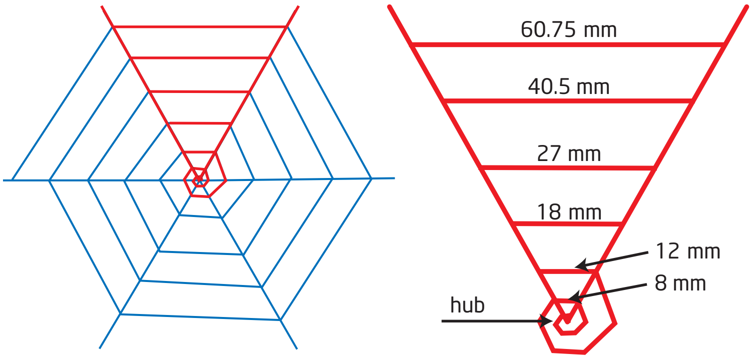
1.3

Geometric Sequences

Focus on...

- providing and justifying an example of a geometric sequence
- deriving a rule for determining the general term of a geometric sequence
- solving a problem that involves a geometric sequence

Many types of sequences can be found in nature. The Fibonacci sequence, frequently found in flowers, seeds, and trees, is one example. A **geometric sequence** can be approximated by the orb web of the common garden spider. A spider's orb web is an impressive architectural feat. The web can capture the beauty of the morning dew, as well as the insects that the spider may feed upon. The following graphic was created to represent an approximation of the geometric sequence formed by the orb web.



The capture spiral is constructed by the spider starting on the outside edge of the web frame, and winding inward toward the hub. The lengths of the sections of the silk between the radii for this section of the spiral produce a geometric sequence. What makes this sequence geometric?

Did You Know?

An orb web is a round spider web with a pattern of lines in a spiral formation.

geometric sequence

- a sequence in which the ratio of consecutive terms is constant

Investigate a Geometric Sequence

Coin Toss Outcomes

Work with a partner for the following activity.

1. a) Toss a single coin. How many possible outcomes are there?

b) Toss two coins. How many possible outcomes are there?

c) Create a tree diagram to show the possible outcomes for three coins.

2. Copy the table. Continue the pattern to complete the table.

Number of Coins, n	Number of Outcomes, t_n	Expanded Form	Using Exponents
1	2	(2)	2^1
2	4	(2)(2)	2^2
3			
4			
⋮	⋮	⋮	⋮
n			

3. a) As the number of coins increases, a sequence is formed by the number of outcomes. What are the first four terms of this sequence?

b) Describe how the terms of the sequence are related. Is this relationship different from an arithmetic sequence? Explain.

c) Predict the next two terms of the sequence. Describe the method you used to make your prediction.

d) Describe a method you could use to generate one term from the previous term.

4. a) For several pairs of consecutive terms in the sequence, divide the second term by the preceding term.

b) What observation can you make about your predictions in step 3c)?

Reflect and Respond

5. a) Is the sequence generated a geometric sequence? How do you know?

b) Write a general term that relates the number of outcomes to the number of coins tossed.

c) Show how to use your formula to determine the value of the 20th term of the sequence.

Materials

- 3 coins

Link the Ideas

common ratio

- the ratio of successive terms in a geometric sequence, $r = \frac{t_n}{t_{n-1}}$
- the ratio may be positive or negative
- for example, in the sequence 2, 4, 8, 16, ..., the common ratio is 2

In a geometric sequence, the ratio of consecutive terms is constant. The **common ratio**, r, can be found by taking any term, except the first, and dividing that term by the preceding term.

The general geometric sequence is $t_1, t_1r, t_1r^2, t_1r^3, \ldots$, where t_1 is the first term and r is the common ratio.

$$\begin{aligned} t_1 &= t_1 \\ t_2 &= t_1r \\ t_3 &= t_1r^2 \\ t_4 &= t_1r^3 \\ &\vdots \\ t_n &= t_1r^{n-1} \end{aligned}$$

The *general term* of a geometric sequence where n is a positive integer is

$$\boldsymbol{t_n = t_1r^{n-1}}$$

where t_1 is the first term of the sequence
n is the number of terms
r is the common ratio
t_n is the general term or nth term

Example 1

Determine t_1, r, and t_n

In nature, many single-celled organisms, such as bacteria, reproduce by splitting in two so that one cell gives rise to 2, then 4, then 8 cells, and so on, producing a geometric sequence. Suppose there were 10 bacteria originally present in a bacteria sample. Determine the general term that relates the number of bacteria to the doubling period of the bacteria. State the values for t_1 and r in the geometric sequence produced.

Did You Know?

One of the most common bacteria on Earth, Shewanella oneidensis MR-1, uses oxygen as an energy source for respiration. This bacterium is generally associated with the removal of metal pollutants in aquatic and marine environments.

Solution

State the sequence generated by the doubling of the bacteria.

$$\begin{aligned} t_1 &= 10 \\ t_2 &= 20 \\ t_3 &= 40 \\ t_4 &= 80 \\ t_5 &= 160 \\ &\vdots \end{aligned}$$

The common ratio, r, may be found by dividing any two consecutive terms, $r = \frac{t_n}{t_{n-1}}$.

$\frac{20}{10} = 2 \quad \frac{40}{20} = 2 \quad \frac{80}{40} = 2 \quad \frac{160}{80} = 2$

The common ratio is 2.

For the given sequence, $t_1 = 10$ and $r = 2$. Use the general term of a geometric sequence.

$t_n = t_1 r^{n-1}$

$t_n = (10)(2)^{n-1}$ Substitute known values.

The general term of the sequence is $t_n = 10(2)^{n-1}$.

Your Turn

Suppose there were three bacteria originally present in a sample. Determine the general term that relates the number of bacteria to the doubling period of the bacteria. State the values of t_1 and r in the geometric sequence formed.

Example 2

Determine a Particular Term

Sometimes you use a photocopier to create enlargements or reductions. Suppose the actual length of a photograph is 25 cm and the smallest size that a copier can make is 67% of the original. What is the shortest possible length of the photograph after 5 reductions? Express your answer to the nearest tenth of a centimetre.

Solution

This situation can be modelled by a geometric sequence.

For this sequence,

First term	$t_1 = 25$
Common ratio	$r = 0.67$
Number of terms	$n = 6$

Why is the number of terms 6 in this case?

You need to find the sixth term of the sequence.

Use the general term, $t_n = t_1 r^{n-1}$.

$t_n = t_1 r^{n-1}$

$t_6 = 25(0.67)^{6-1}$ Substitute known values.

$t_6 = 25(0.67)^5$

$t_6 = 3.375\ldots$

After five reductions, the shortest possible length of the photograph is approximately 3.4 cm.

Your Turn

Suppose the smallest reduction a photocopier could make is 60% of the original. What is the shortest possible length after 8 reductions of a photograph that is originally 42 cm long?

Example 3

Determine t_1 and r

In a geometric sequence, the third term is 54 and the sixth term is -1458. Determine the values of t_1 and r, and list the first three terms of the sequence.

Solution

Method 1: Use Logical Reasoning

The third term of the sequence is 54 and the sixth term is -1458.

$t_3 = 54$
$t_6 = -1458$

Since the sequence is geometric,

$$\begin{aligned} t_4 &= t_3(r) \\ t_5 &= t_3(r)(r) \\ t_6 &= t_3(r)(r)(r) && \text{Substitute known values.} \\ -1458 &= 54r^3 \\ \frac{-1458}{54} &= r^3 \\ -27 &= r^3 \\ \sqrt[3]{-27} &= r \\ -3 &= r \end{aligned}$$

You can use the general term of a geometric sequence to determine the value for t_1.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ t_3 &= t_1r^{3-1} \\ t_3 &= t_1r^2 \\ 54 &= t_1(-3)^2 && \text{Substitute known values.} \\ 54 &= 9t_1 \\ 6 &= t_1 \end{aligned}$$

The first term of the sequence is 6 and the common ratio is -3.
The first three terms of the sequence are 6, -18, 54.

Method 2: Use the General Term

You can write an equation for t_3 and an equation for t_6 using the general term of a geometric sequence.

$t_n = t_1r^{n-1}$

For the third term, $n = 3$.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ 54 &= t_1r^{3-1} \\ 54 &= t_1r^2 \end{aligned}$$

For the sixth term, $n = 6$.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ -1458 &= t_1r^{6-1} \\ -1458 &= t_1r^5 \end{aligned}$$

Solve one of the equations for the variable t_1.

$$54 = t_1r^2$$
$$\frac{54}{r^2} = t_1$$

Substitute this expression for t_1 in the other equation. Solve for the variable r.

$$-1458 = t_1r^5$$
$$-1458 = \left(\frac{54}{r^2}\right)r^5$$
$$-1458 = 54r^3$$
$$\frac{-1458}{54} = \frac{54r^3}{54}$$
$$-27 = r^3$$
$$\sqrt[3]{-27} = r$$
$$-3 = r$$

Substitute the common ratio of -3 in one of the equations to solve for the first term, t_1.

Substitute $r = -3$

$$54 = t_1r^2$$
$$54 = t_1(-3)^2$$
$$54 = 9t_1$$
$$6 = t_1$$

The first term of the sequence is 6 and the common ratio is -3.
The first three terms of the sequence are 6, -18, 54.

Your Turn

In a geometric sequence, the second term is 28 and the fifth term is 1792. Determine the values of t_1 and r, and list the first three terms of the sequence.

Example 4

Apply Geometric Sequences

The modern piano has 88 keys. The frequency of the notes ranges from A_0, the lowest note, at 27.5 Hz, to C_8, the highest note on the piano, at 4186.009 Hz. The frequencies of these notes approximate a geometric sequence as you move up the keyboard.

a) Determine the common ratio of the geometric sequence produced from the lowest key, A_0, to the fourth key, C_1, at 32.7 Hz.

b) Use the lowest and highest frequencies to verify the common ratio found in part a).

Did You Know?

A sound has two characteristics, pitch and volume. The pitch corresponds to the frequency of the sound wave. High notes have high frequencies. Low notes have low frequencies. Frequency is measured in Hertz (Hz), which is the number of waves per second.

Solution

a) The situation may be modelled by a geometric sequence.
For this sequence,

First term	$t_1 = 27.5$
Number of terms	$n = 4$
*n*th term	$t_n = 32.7$

Use the general term of a geometric sequence.

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ 32.7 &= (27.5)(r^{4-1}) && \text{Substitute known values.} \\ \frac{32.7}{27.5} &= \frac{27.5r^3}{27.5} \\ \frac{32.7}{27.5} &= r^3 \\ \sqrt[3]{\frac{32.7}{27.5}} &= r && \text{Take the cube root of both sides.} \\ 1.0594\ldots &= r \end{aligned}$$

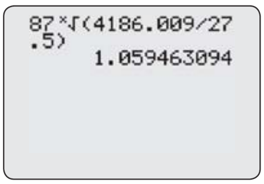
The common ratio for this sequence is approximately 1.06.

b) For this sequence,

First term	$t_1 = 27.5$
Number of terms	$n = 88$
*n*th term	$t_n = 4186.009$

Use the general term of a geometric sequence.

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ 4186.009 &= (27.5)(r^{88-1}) && \text{Substitute known values.} \\ \frac{4186.009}{27.5} &= \frac{27.5r^{87}}{27.5} \\ \frac{4186.009}{27.5} &= r^{87} \\ \sqrt[87]{\frac{4186.009}{27.5}} &= r && \text{Take the 87th root of both sides.} \\ 1.0594\ldots &= r \end{aligned}$$



The common ratio of this sequence is approximately 1.06.

Your Turn

In 1990 the population of Canada was approximately 26.6 million. The population projection for 2025 is approximately 38.4 million. If this projection were based on a geometric sequence, what would be the annual growth rate? Given that this is a geometric sequence what assumptions would you have to make?

Key Ideas

- A geometric sequence is a sequence in which each term, after the first term, is found by multiplying the previous term by a non-zero constant, r, called the common ratio.
- The common ratio of successive terms of a geometric sequence can be found by dividing any two consecutive terms, $r = \frac{t_n}{t_{n-1}}$.
- The general term of a geometric sequence is
  $t_n = t_1 r^{n-1}$
  where t_1 is the first term
  n is the number of terms
  r is the common ratio
  t_n is the general term or nth term

Check Your Understanding

Practise

1. Determine if the sequence is geometric. If it is, state the common ratio and the general term in the form $t_n = t_1 r^{n-1}$.

a) 1, 2, 4, 8, …

b) 2, 4, 6, 8, …

c) 3, −9, 27, −81, …

d) 1, 1, 2, 4, 8, …

e) 10, 15, 22.5, 33.75, …

f) −1, −5, −25, −125, …

2. Copy and complete the following table for the given geometric sequences.

	Geometric Sequence	Common Ratio	6th Term	10th Term
a)	6, 18, 54, ...			
b)	1.28, 0.64, 0.32, ...			
c)	$\frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \ldots$			

3. Determine the first four terms of each geometric sequence.

a) $t_1 = 2, r = 3$

b) $t_1 = -3, r = -4$

c) $t_1 = 4, r = -3$

d) $t_1 = 2, r = 0.5$

4. Determine the missing terms, t_2, t_3, and t_4, in the geometric sequence in which $t_1 = 8.1$ and $t_5 = 240.1$.

5. Determine a formula for the nth term of each geometric sequence.

a) $r = 2, t_1 = 3$

b) 192, −48, 12, −3, …

c) $t_3 = 5, t_6 = 135$

d) $t_1 = 4, t_{13} = 16\ 384$

Apply

6. Given the following geometric sequences, determine the number of terms, n.

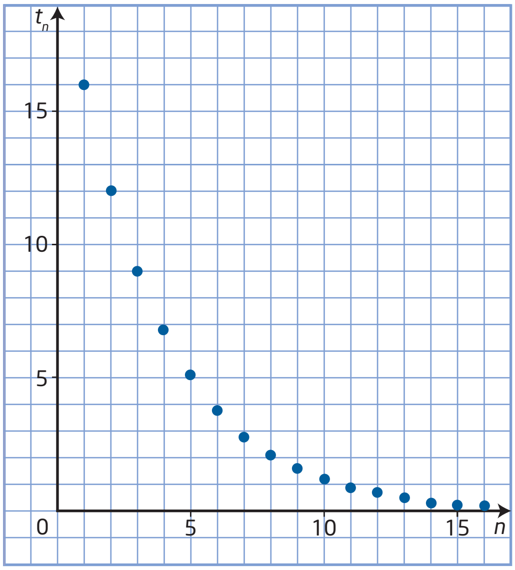
Table A

	First Term, t_1	Common Ratio, r	nth Term, t_n	Number of Terms, n
a)	5	3	135	
b)	−2	−3	−1458	
c)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{48}$	
d)	4	4	4096	
e)	$-\frac{1}{6}$	2	$-\frac{128}{3}$	
f)	$\frac{p^2}{2}$	$\frac{p}{2}$	$\frac{p^9}{256}$	

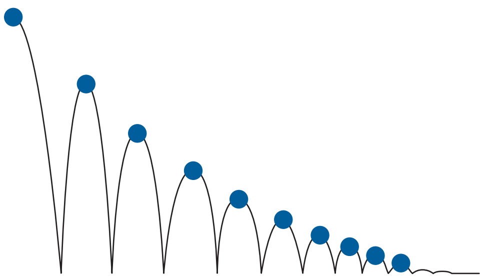
7. The following sequence is geometric. What is the value of y?

$3, 12, 48, 5y + 7, \ldots$

8. The following graph illustrates a geometric sequence. List the first three terms for the sequence and state the general term that describes the sequence.



9. A ball is dropped from a height of 3.0 m. After each bounce it rises to 75% of its previous height.



a) Write the first term and the common ratio of the geometric sequence.

b) Write the general term for the sequence in part a).

c) What height does the ball reach after the 6th bounce?

d) After how many bounces will the ball reach a height of approximately 40 cm?

10. The colour of some clothing fades over time when washed. Suppose a pair of jeans fades by 5% with each washing.

a) What percent of the colour remains after one washing?

b) If $t_1 = 100$, what are the first four terms of the sequence?

c) What is the value of r for your geometric sequence?

d) What percent of the colour remains after 10 washings?

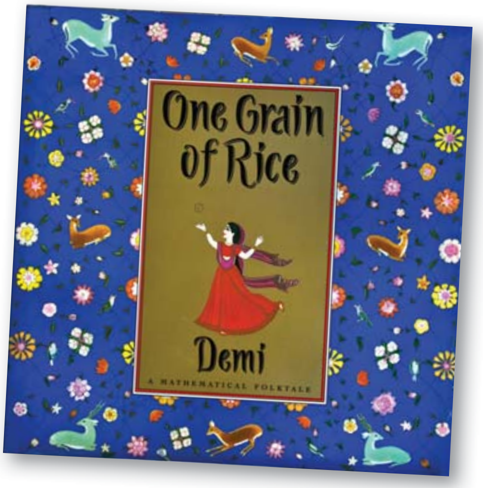
e) How many washings would it take so that only 25% of the original colour remains in the jeans? What assumptions did you make?

11. Pincher Creek, in the foothills of the Rocky Mountains in southern Alberta, is an ideal location to harness the wind power of the chinook winds that blow through the mountain passes. Kinetic energy from the moving air is converted to electricity by wind turbines. In 2004, the turbines generated 326 MW of wind energy, and it is projected that the amount will be 10 000 MW per year by 2010. If this growth were modelled by a geometric sequence, determine the value of the annual growth rate from 2004 to 2010.

Did You Know?

In an average year, a single 660-kW wind turbine produces 2000 MW of electricity, enough power for over 250 Canadian homes. Using wind to produce electricity rather than burning coal will leave 900 000 kg of coal in the ground and emit 2000 tonnes fewer greenhouse gases annually. This has the same positive impact as taking 417 cars off the road or planting 10 000 trees.

12. The following excerpt is taken from the book *One Grain of Rice* by Demi.



Long ago in India, there lived a raja who believed that he was wise and fair. But every year he kept nearly all of the people's rice for himself. Then when famine came, the raja refused to share the rice, and the people went hungry. Then a village girl named Rani devises a clever plan. She does a good deed for the raja, and in return, the raja lets her choose her reward. Rani asks for just one grain of rice, doubled every day for thirty days.

a) Write the sequence of terms for the first five days that Rani would receive the rice.

b) Write the general term that relates the number of grains of rice to the number of days.

c) Use the general term to determine the number of grains of rice that Rani would receive on the 30th day.

13. The Franco-Manitoban community of St-Pierre-Jolys celebrates Les Folies Grenouilles annually in August. Some of the featured activities include a slow pitch tournament, a parade, fireworks, and the Canadian National Frog Jumping Championships. During the competition, competitor's frogs have five chances to reach their maximum jump. One year, a frog by the name of Georges, achieved the winning jump in his 5th try. Georges' first jump was 191.41 cm, his second jump was 197.34 cm, and his third was 203.46 cm. The pattern of Georges' jumps approximated a geometric sequence.

a) By what ratio did Georges improve his performance with each jump? Express your answer to three decimal places.

b) How far was Georges' winning jump? Express your answer to the nearest tenth of a centimetre.

c) The world record frog jump is held by a frog named Santjie of South Africa. Santjie jumped approximately 10.2 m. If Georges, from St-Pierre-Jolys, had continued to increase his jumps following this same geometric sequence, how many jumps would Georges have needed to complete to beat Santjie's world record jump?

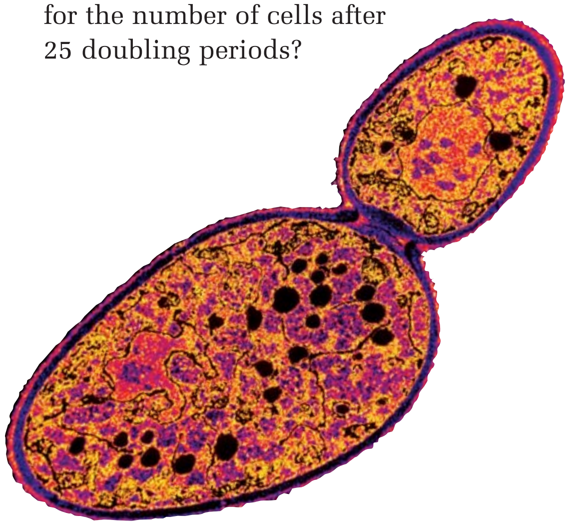
14. Bread and bread products have been part of our diet for centuries. To help bread rise, yeast is added to the dough. Yeast is a living unicellular micro-organism about one hundredth of a millimetre in size. Yeast multiplies by a biochemical process called budding. After mitosis and cell division, one cell results in two cells with exactly the same characteristics.

a) Write a sequence for the first six terms that describes the cell growth of yeast, beginning with a single cell.

b) Write the general term for the growth of yeast.

c) How many cells would there be after 25 doublings?

d) What assumptions would you make for the number of cells after 25 doubling periods?



15. The Arctic Winter Games is a high profile sports competition for northern and arctic athletes. The premier sports are the Dene and Inuit games, which include the arm pull, the one foot high kick, the two foot high kick, and the Dene hand games. The games are held every two years. The first Arctic Winter Games, held in 1970, drew 700 competitors. In 2008, the games were held in Yellowknife and drew 2000 competitors. If the number of competitors grew geometrically from 1970 to 2008, determine the annual rate of growth in the number of competitors from one Arctic Winter Games to the next. Express your answer to the nearest tenth of a percent.

16. Jason Annahatak entered the Russian sledge jump competition at the Arctic Winter Games, held in Yellowknife. Suppose that to prepare for this event, Jason started training by jumping 2 sledges each day for the first week, 4 sledges each day for the second week, 8 sledges each day for the third week, and so on. During the competition, Jason jumped 142 sledges. Assuming he continued his training pattern, how many weeks did it take him to reach his competition number of 142 sledges?

Did You Know?

Sledge jump starts from a standing position. The athlete jumps consecutively over 10 sledges placed in a row, turns around using one jumping movement, and then jumps back over the 10 sledges. This process is repeated until the athlete misses a jump or touches a sledge.

17. At Galaxyland in the West Edmonton Mall, a boat swing ride has been modelled after a basic pendulum design. When the boat first reaches the top of the swing, this is considered to be the beginning of the first swing. A swing is completed when the boat changes direction. On each successive completed swing, the boat travels 96% as far as on the previous swing. The ride finishes when the arc length through which the boat travels is 30 m. If it takes 20 swings for the boat to reach this arc length, determine the arc length through which the boat travels on the first swing. Express your answer to the nearest tenth of a metre.

18. The Russian nesting doll or Matryoshka had its beginnings in 1890. The dolls are made so that the smallest doll fits inside a larger one, which fits inside a larger one, and so on, until all the dolls are hidden inside the largest doll. In a set of 50 dolls, the tallest doll is 60 cm and the smallest is 1 cm. If the decrease in doll size approximates a geometric sequence, determine the common ratio. Express your answer to three decimal places.

19. The primary function for our kidneys is to filter our blood to remove any impurities. Doctors take this into account when prescribing the dosage and frequency of medicine. A person's kidneys filter out 18% of a particular medicine every two hours.

a) How much of the medicine remains after 12 h if the initial dosage was 250 mL? Express your answer to the nearest tenth of a millilitre.

b) When there is less than 20 mL left in the body, the medicine becomes ineffective and another dosage is needed. After how many hours would this happen?

Did You Know?

Every day, a person's kidneys process about 190 L of blood to remove about 1.9 L of waste products and extra water.

20. The charge in a car battery, when the car is left to sit, decreases by about 2% per day and can be modelled by the formula $C = 100(0.98)^d$, where d is the time, in days, and C is the approximate level of charge, as a percent.

a) Copy and complete the chart to show the percent of charge remaining in relation to the time passed.

Time, d (days)	Charge Level, C (%)
0	100
1	
2	
3	

b) Write the general term of this geometric sequence.

c) Explain how this formula is different from the formula $C = 100(0.98)^d$.

d) How much charge is left after 10 days?

21. A coiled basket is made using dried pine needles and sinew. The basket is started from the centre using a small twist and spirals outward and upward to shape the basket. The circular coiling of the basket approximates a geometric sequence, where the radius of the first coil is 6 mm.

a) If the ratio of consecutive coils is 1.22, calculate the radius for the 8th coil.

b) If there are 18 coils, what is the circumference of the top coil of the basket?

Extend

22. Demonstrate that 6^a, 6^b, 6^c, … forms a geometric sequence when a, b, c, … forms an arithmetic sequence.

23. If $x + 2$, $2x + 1$, and $4x - 3$ are three consecutive terms of a geometric sequence, determine the value of the common ratio and the three given terms.

24. On a six-string guitar, the distance from the nut to the bridge is 38 cm. The distance from the first fret to the bridge is 35.87 cm, and the distance from the second fret to the bridge is 33.86 cm. This pattern approximates a geometric sequence.

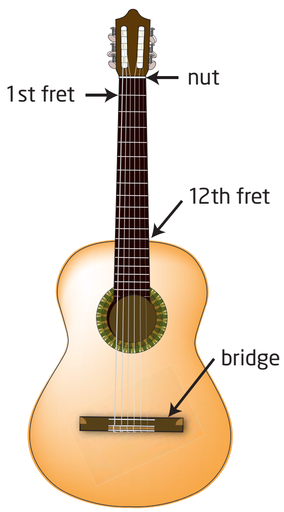
a) What is the distance from the 8th fret to the bridge?

b) What is the distance from the 12th fret to the bridge?

c) Determine the distance from the nut to the first fret.

d) Determine the distance from the first fret to the second fret.

e) Write the sequence for the first three terms of the distances between the frets. Is this sequence geometric or arithmetic? What is the common ratio or common difference?



Create Connections

25. Alex, Mala, and Paul were given the following problem to solve in class.

An aquarium that originally contains 40 L of water loses 8% of its water to evaporation every day. Determine how much water will be in the aquarium at the beginning of the 7th day.

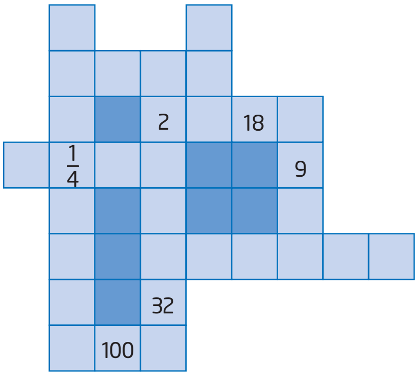
The three students' solutions are shown below. Which approach to the solution is correct? Justify your reasoning.

Alex's solution:
Alex believed that the sequence was geometric, where $t_1 = 40$, $r = 0.08$, and $n = 7$. He used the general formula $t_n = t_1r^{n-1}$.
$t_n = t_1r^{n-1}$
$t_n = 40(0.08)^{n-1}$
$t_7 = 40(0.08)^{7-1}$
$t_7 = 40(0.08)^6$
$t_7 = 0.000\ 01$
There will be 0.000 01 L of water in the tank at the beginning of the 7th day.

Mala's solution:
Mala believed that the sequence was geometric, where $t_1 = 40$, $r = 0.92$, and $n = 7$. She used the general formula $t_n = t_1r^{n-1}$.
$t_n = t_1r^{n-1}$
$t_n = 40(0.92)^{n-1}$
$t_7 = 40(0.92)^{7-1}$
$t_7 = 40(0.92)^6$
$t_7 = 24.25$
There will be 24.25 L of water in the tank at the beginning of the 7th day.

Paul's solution:
Paul believed that the sequence was arithmetic, where $t_1 = 40$ and $n = 7$. To calculate the value of d, Paul took 8% of 40 = 3.2. He reasoned that this would be a negative constant since the water was gradually disappearing. He used the general formula $t_n = t_1 + (n - 1)d$.
$t_n = t_1 + (n - 1)d$
$t_n = 40 + (n - 1)(-3.2)$
$t_7 = 40 + (7 - 1)(-3.2)$
$t_7 = 40 + (6)(-3.2)$
$t_7 = 20.8$
There will be 20.8 L of water in the tank at the beginning of the 7th day.

26. Copy the puzzle. Fill in the empty boxes with positive numbers so that each row and column forms a geometric sequence.



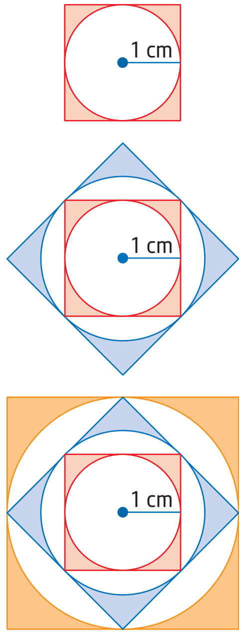
27. A square has an inscribed circle of radius 1 cm.

a) What is the area of the red portion of the square, to the nearest hundredth of a square centimetre?

b) If another square with an inscribed circle is drawn around the original, what is the area of the blue region, to the nearest hundredth of a square centimetre?

c) If another square with an inscribed circle is drawn around the squares, what is the area of the orange region, to the nearest hundredth of a square centimetre?

d) If this pattern were to continue, what would be the area of the newly coloured region for the 8th square, to the nearest hundredth of a square centimetre?



Project Corner — Forestry

- Canada has 402.1 million hectares (ha) of forest and other wooded lands. This value represents 41.1% of Canada's total surface area of 979.1 million hectares.
- Annually, Canada harvests 0.3% of its commercial forest area. In 2007, 0.9 million hectares were harvested.
- In 2008, British Columbia planted its 6 billionth tree seedling since the 1930s, as part of its reforestation programs.

1.4

Geometric Series

Focus on…

- deriving a rule for determining the sum of n terms of a geometric series
- determining t_1, r, n, or S_n involving a geometric series
- solving a problem that involves a geometric series
- identifying any assumptions made when identifying a geometric series

If you take the time to look closely at nature, chances are you have seen a fractal. Fractal geometry is the *geometry of nature*. The study of fractals is, mathematically, relatively new. A fractal is a geometric figure that is generated by starting with a very simple pattern and repeating that pattern over and over an infinite number of times. The basic concept of a fractal is that it contains a large degree of self-similarity. This means that a fractal usually contains small copies of itself buried within the original. Where do you see fractals in the images shown?

Investigate Fractals

Materials

- paper
- ruler

Fractal Tree

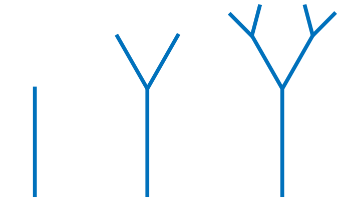
A *fractal tree* is a fractal pattern that results in a realistic looking tree.

You can build your own fractal tree:

1. a) Begin with a sheet of paper. Near the bottom of the paper and centred on the page, draw a vertical line segment approximately 3 cm to 4 cm in length.

b) At the top of the segment, draw two line segments, splitting away from each other as shown in Stage 2. These segments form the branches of the tree. Each new branch formed is a smaller version of the main trunk of the tree.

c) At the top of each new line segment, draw another two branches, as shown in Stage 3.



d) Continue this process to complete five stages of the fractal tree.

2. Copy and complete the following table.

Stage	1	2	3	4	5
Number of New Branches	1	2			

3. Decide whether a geometric sequence has been generated for the number of new branches formed at each stage. If a geometric sequence has been generated, state the first term, the common ratio, and the general term.

Reflect and Respond

4. a) Would a geometric sequence be generated if there were three new branches formed from the end of each previous branch?

b) Would a geometric sequence be generated if there were four new branches formed?

5. Describe a strategy you could use to determine the total number of branches that would be formed by the end of stage 5.

6. Would this be a suitable strategy to use if you wanted to determine the total number of branches up to stage 100? Explain.

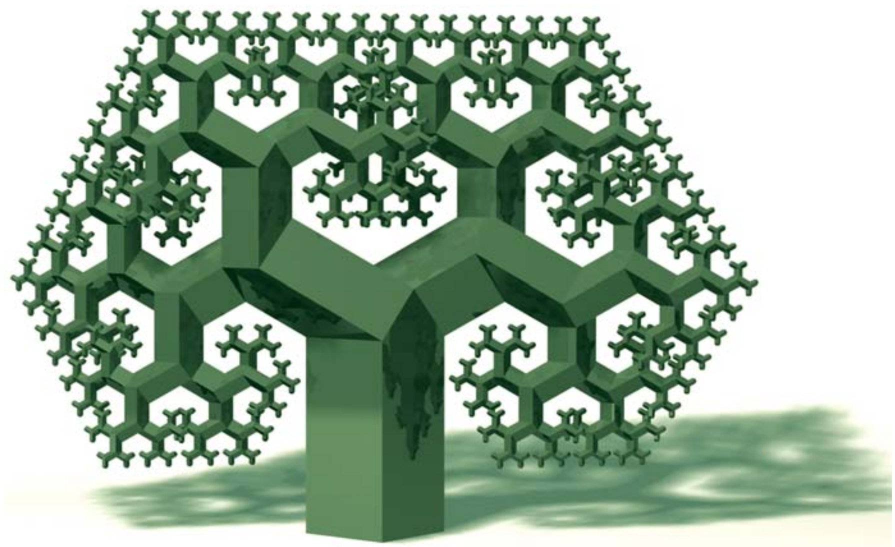


Image rendered by Anton Bakker based on a fractal tree design by Koos Verhoeff. Used with permission of the Foundation MathArt Koos Verhoeff.

Web Link

The complex mathematical equations of fractals are used in the creation of many works of art and computer generated fractals. To learn more about art and fractals, go to www.mhrprecalc11.ca and follow the links.

Link the Ideas

geometric series

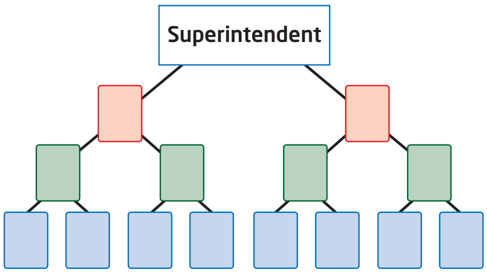
- the terms of a geometric sequence expressed as a sum
- for example, $3 + 6 + 12 + 24$ is a geometric series

A **geometric series** is the expression for the sum of the terms of a geometric sequence.

A school district emergency *fan-out* system is designed to enable important information to reach the entire staff of the district very quickly. At the first level, the superintendent calls two assistant superintendents. The two assistant superintendents each call two area superintendents. They in turn, each call two principals. The pattern continues with each person calling two other people.

At every level, the total number of people contacted is twice the number of people contacted in the previous level. The pattern can be modelled by a geometric series where the first term is 1 and the common ratio is 2. The series for the fan-out system would be $1 + 2 + 4 + 8$, which gives a sum of 15 people contacted after 4 levels.

To extend this series to 15 or 20 or 100 levels, you need to determine a way to calculate the sum of the series other than just adding the terms.



One way to calculate the sum of the series is to use a formula.

To develop a formula for the sum of a series,

List the original series.
$S_4 = 1 + 2 + 4 + 8$ ①

Multiply each term in the series by the common ratio.
$2(S_4 = 1 + 2 + 4 + 8)$
$2S_4 = 2 + 4 + 8 + 16$ ②

The number of staff contacted in the 5th level is 16.

Subtract equation ① from equation ②.

$$\begin{array}{rrl} 2S_4 = & & 2 + 4 + 8 + 16 \\ -\ S_4 = & 1 + & 2 + 4 + 8 \\ \hline (2 - 1)S_4 = & -1 + & 0 + 0 + 0 + 16 \end{array}$$

Why are the two equations aligned as shown?

Isolate S_4 by dividing by $(2 - 1)$.

$S_4 = \dfrac{16 - 1}{2 - 1}$

$S_4 = 15$

You can use the above method to derive a general formula for the sum of a geometric series.

The general geometric series may be represented by the following series.

$S_n = t_1 + t_1r + t_1r^2 + t_1r^3 + \cdots + t_1r^{n-1}$

Multiply every term in the series by the common ratio, r.

$rS_n = t_1r + t_1r(r) + t_1r^2(r) + t_1r^3(r) + \cdots + t_1r^{n-1}(r)$

$rS_n = t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \cdots + t_1r^n$

Subtract the two equations.

$$\begin{array}{rcl} rS_n &=& \phantom{t_1 +{}} t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \cdots + t_1r^{n-1} + t_1r^n \\ S_n &=& t_1 + t_1r + t_1r^2 + t_1r^3 + \cdots + t_1r^{n-1} \\ \hline (r-1)S_n &=& -t_1 + 0 + 0 + 0 + \cdots + 0 + 0 + t_1r^n \end{array}$$

Isolate S_n by dividing by $r - 1$.

$S_n = \dfrac{t_1r^n - t_1}{r-1}$ or $S_n = \dfrac{t_1(r^n - 1)}{r-1}$, $r \neq 1$

Why can r not be equal to 1?

The *sum* of a geometric series can be determined using the formula

$$\boldsymbol{S_n = \frac{t_1(r^n - 1)}{r-1}, r \neq 1}$$

where t_1 is the first term of the series
n is the number of terms
r is the common ratio
S_n is the sum of the first n terms

Example 1

Determine the Sum of a Geometric Series

Determine the sum of the first 10 terms of each geometric series.

a) $4 + 12 + 36 + \cdots$

b) $t_1 = 5$, $r = \dfrac{1}{2}$

Solution

a) In the series, $t_1 = 4$, $r = 3$, and $n = 10$.

$$S_n = \frac{t_1(r^n - 1)}{r-1}$$

$$S_{10} = \frac{4(3^{10} - 1)}{3-1}$$

$$S_{10} = \frac{4(59\,048)}{2}$$

$$S_{10} = 118\,096$$

The sum of the first 10 terms of the geometric series is 118 096.

b) In the series, $t_1 = 5$, $r = \frac{1}{2}$, $n = 10$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{5\left[\left(\frac{1}{2}\right)^{10} - 1\right]}{\frac{1}{2} - 1}$$

$$S_{10} = \frac{5\left(\frac{1}{1024} - 1\right)}{-\frac{1}{2}}$$

$$S_{10} = -10\left(\frac{-1023}{1024}\right)$$

$$S_{10} = \frac{5115}{512}$$

The sum of the first 10 terms of the geometric series is $\frac{5115}{512}$ or $9\frac{507}{512}$.

Your Turn

Determine the sum of the first 8 terms of the following geometric series.

a) $5 + 15 + 45 + \cdots$

b) $t_1 = 64$, $r = \frac{1}{4}$

Example 2

Determine the Sum of a Geometric Series for an Unspecified Number of Terms

Determine the sum of each geometric series.

a) $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \cdots + 729$

b) $4 - 16 + 64 - \cdots - 65\ 536$

Solution

a) Method 1: Determine the Number of Terms

$$t_n = t_1 r^{n-1}$$ Use the general term.

$$729 = \frac{1}{27}(3)^{n-1}$$ Substitute known values.

$$(27)(729) = \left[\frac{1}{27}(3)^{n-1}\right](27)$$ Multiply both sides by 27.

$$(27)(729) = (3)^{n-1}$$

$$(3^3)(3^6) = (3)^{n-1}$$ Write as powers with a base of 3.

$$(3)^9 = (3)^{n-1}$$

$$9 = n - 1$$

$$10 = n$$ Since the bases are the same, the exponents must be equal.

There are 10 terms in the series.

Use the general formula for the sum of a geometric series where $n = 10$, $t_1 = \frac{1}{27}$, and $r = 3$.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{\left(\frac{1}{27}\right)[(3)^{10} - 1]}{3 - 1}$$

$$S_{10} = \frac{29\,524}{27}$$

The sum of the series is $\frac{29\,524}{27}$ or $1093\frac{13}{27}$.

Method 2: Use an Alternate Formula

Begin with the formula for the general term of a geometric sequence,
$t_n = t_1 r^{n-1}$.

Multiply both sides by r.

$rt_n = (t_1 r^{n-1})(r)$

Simplify the right-hand side of the equation.

$rt_n = t_1 r^n$

From the previous work, you know that the general formula for the sum of a geometric series may be written as

$$S_n = \frac{t_1 r^n - t_1}{r - 1}$$

Substitute rt_n for $t_1 r^n$.

$$S_n = \frac{rt_n - t_1}{r - 1} \text{ where } r \neq 1.$$

This results in a general formula for the sum of a geometric series when the first term, the nth term, and the common ratio are known.

Determine the sum where $r = 3$, $t_n = 729$, and $t_1 = \frac{1}{27}$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{(3)(729) - \frac{1}{27}}{3 - 1}$$

$$S_n = \frac{29\,524}{27}$$

The sum of the series is $\frac{29\,524}{27}$ or approximately 1093.48.

b) Use the alternate formula $S_n = \frac{rt_n - t_1}{r - 1}$, where $t_1 = 4$, $r = -4$, and $t_n = -65\,536$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{(-4)(-65\,536) - 4}{-4 - 1}$$

$$S_n = -52\,428$$

The sum of the series is $-52\,428$.

Your Turn

Determine the sum of the following geometric series.

a) $\frac{1}{64} + \frac{1}{16} + \frac{1}{4} + \cdots + 1024$

b) $-2 + 4 - 8 + \cdots - 8192$

Example 3

Apply Geometric Series

The Western Scrabble™ Network is an organization whose goal is to promote the game of Scrabble™. It offers Internet tournaments throughout the year that WSN members participate in. The format of these tournaments is such that the losers of each round are eliminated from the next round. The winners continue to play until a final match determines the champion. If there are 256 entries in an Internet Scrabble™ tournament, what is the total number of matches that will be played in the tournament?

Solution

The number of matches played at each stage of the tournament models the terms of a geometric sequence. There are two players per match, so the first term, t_1, is $\frac{256}{2} = 128$ matches. After the first round, half of the players are eliminated due to a loss. The common ratio, r, is $\frac{1}{2}$.

A single match is played at the end of the tournament to decide the winner. The nth term of the series, t_n, is 1 final match.

Use the formula $S_n = \frac{rt_n - t_1}{r - 1}$ for the sum of a geometric series where $t_1 = 128$, $r = \frac{1}{2}$, and $t_n = 1$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{\left(\frac{1}{2}\right)(1) - 128}{\left(\frac{1}{2}\right) - 1}$$

$$S_n = \frac{\frac{-255}{2}}{-\frac{1}{2}}$$

$$S_n = \left(\frac{-255}{2}\right)\left(-\frac{2}{1}\right)$$

$$S_n = 255$$

There will be 255 matches played in the tournament

Your Turn

If a tournament has 512 participants, how many matches will be played?

Key Ideas

- A geometric series is the expression for the sum of the terms of a geometric sequence.
  For example, $5 + 10 + 20 + 40 + \cdots$ is a geometric series.
- The general formula for the sum of the first n terms of a geometric series with the first term, t_1, and the common ratio, r, is
  $S_n = \dfrac{t_1(r^n - 1)}{r - 1}$, $r \neq 1$
- A variation of this formula may be used when the first term, t_1, the common ratio, r, and the nth term, t_n, are known, but the number of terms, n, is not known.
  $S_n = \dfrac{rt_n - t_1}{r - 1}$, $r \neq 1$

Check Your Understanding

Practise

1. Determine whether each series is geometric. Justify your answer.

a) $4 + 24 + 144 + 864 + \cdots$

b) $-40 + 20 - 10 + 5 - \cdots$

c) $3 + 9 + 18 + 54 + \cdots$

d) $10 + 11 + 12.1 + 13.31 + \cdots$

2. For each geometric series, state the values of t_1 and r. Then determine each indicated sum. Express your answers as exact values in fraction form and to the nearest hundredth.

a) $6 + 9 + 13.5 + \cdots$ (S_{10})

b) $18 - 9 + 4.5 + \cdots$ (S_{12})

c) $2.1 + 4.2 + 8.4 + \cdots$ (S_9)

d) $0.3 + 0.003 + 0.000\ 03 + \cdots$ (S_{12})

3. What is S_n for each geometric series described? Express your answers as exact values in fraction form.

a) $t_1 = 12$, $r = 2$, $n = 10$

b) $t_1 = 27$, $r = \frac{1}{3}$, $n = 8$

c) $t_1 = \frac{1}{256}$, $r = -4$, $n = 10$

d) $t_1 = 72$, $r = \frac{1}{2}$, $n = 12$

4. Determine S_n for each geometric series. Express your answers to the nearest hundredth, if necessary.

a) $27 + 9 + 3 + \cdots + \frac{1}{243}$

b) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{128}{6561}$

c) $t_1 = 5$, $t_n = 81\ 920$, $r = 4$

d) $t_1 = 3$, $t_n = 46\ 875$, $r = -5$

5. What is the value of the first term for each geometric series described? Express your answers to the nearest tenth, if necessary.

a) $S_n = 33$, $t_n = 48$, $r = -2$

b) $S_n = 443$, $n = 6$, $r = \frac{1}{3}$

6. The sum of $4 + 12 + 36 + 108 + \cdots + t_n$ is 4372. How many terms are in the series?

7. The common ratio of a geometric series is $\frac{1}{3}$ and the sum of the first 5 terms is 121.

a) What is the value of the first term?

b) Write the first 5 terms of the series.

8. What is the second term of a geometric series in which the third term is $\frac{9}{4}$ and the sixth term is $-\frac{16}{81}$? Determine the sum of the first 6 terms. Express your answer to the nearest tenth.

Apply

9. A fan-out system is used to contact a large group of people. The person in charge of the contact committee relays the information to four people. Each of these four people notifies four more people, who in turn each notify four more people, and so on.

a) Write the corresponding series for the number of people contacted.

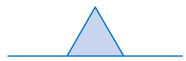
b) How many people are notified after 10 levels of this system?

10. A tennis ball dropped from a height of 20 m bounces to 40% of its previous height on each bounce. The total vertical distance travelled is made up of upward bounces and downward drops. Draw a diagram to represent this situation. What is the total vertical distance the ball has travelled when it hits the floor for the sixth time? Express your answer to the nearest tenth of a metre.

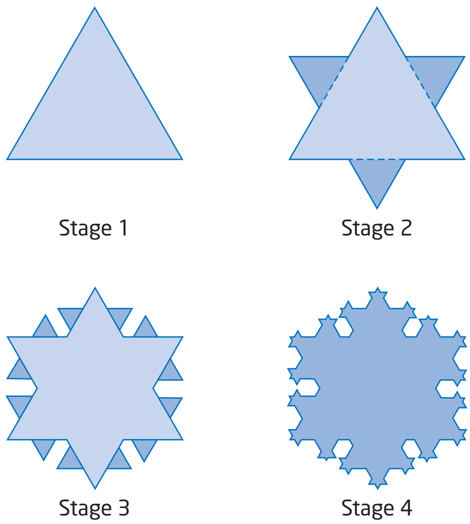
11. Celia is training to run a marathon. In the first week she runs 25 km and increases this distance by 10% each week. This situation may be modelled by the series $25 + 25(1.1) + 25(1.1)^2 + \cdots$. She wishes to continue this pattern for 15 weeks. How far will she have run in total when she completes the 15th week? Express your answer to the nearest tenth of a kilometre.

12. **MINI LAB** Building the Koch snowflake is a step-by-step process.

- Start with an equilateral triangle. (Stage 1)
- In the middle of each line segment forming the sides of the triangle, construct an equilateral triangle with side length equal to $\frac{1}{3}$ of the length of the line segment.



- Delete the base of this new triangle. (Stage 2)
- For each line segment in Stage 2, construct an equilateral triangle, deleting its base. (Stage 3)
- Repeat this process for each line segment, as you move from one stage to the next.



a) Work with a partner. Use dot paper to draw three stages of the Koch snowflake.

b) Copy and complete the following table.

Stage Number	Length of Each Line Segment	Number of Line Segments	Perimeter of Snowflake
1	1	3	3
2	$\frac{1}{3}$	12	4
3	$\frac{1}{9}$		
4			
5			

c) Determine the general term for the length of each line segment, the number of line segments, and the perimeter of the snowflake.

d) What is the total perimeter of the snowflake up to Stage 6?

13. An advertising company designs a campaign to introduce a new product to a metropolitan area. The company determines that 1000 people are aware of the product at the beginning of the campaign. The number of new people aware increases by 40% every 10 days during the advertising campaign. Determine the total number of people who will be aware of the product after 100 days.

14. Bead working has a long history among Canada's Indigenous peoples. Floral designs are the predominate patterns found among people of the boreal forests and northern plains. Geometric patterns are found predominately in the Great Plains. As bead work continues to be popular, traditional patterns are being exchanged among people in all regions. Suppose a set of 10 beads were laid in a line where each successive bead had a diameter that was $\frac{3}{4}$ of the diameter of the previous bead. If the first bead had a diameter of 24 mm, determine the total length of the line of beads. Express your answer to the nearest millimetre.

Did You Know?

Wampum belts consist of rows of beads woven together. Weaving traditionally involves stringing the beads onto twisted plant fibres, and then securing them to animal sinew.

Wanuskewin Native Heritage Park, Cree Nation, Saskatchewan

15. When doctors prescribe medicine at equally spaced time intervals, they are aware that the body metabolizes the drug gradually. After some period of time, only a certain percent of the original amount remains. After each dose, the amount of the drug in the body is equal to the amount of the given dose plus the amount remaining from the previous doses. The amount of the drug present in the body after the nth dose is modelled by a geometric series where t_1 is the prescribed dosage and r is the previous dose remaining in the body.

    Suppose a person with an ear infection takes a 200-mg ampicillin tablet every 4 h. About 12% of the drug in the body at the start of a four-hour period is still present at the end of that period. What amount of ampicillin is in the body, to the nearest tenth of a milligram,

    a) after taking the third tablet?

    b) after taking the sixth tablet?

Extend

16. Determine the number of terms, n, if $3 + 3^2 + 3^3 + \cdots + 3^n = 9840$.

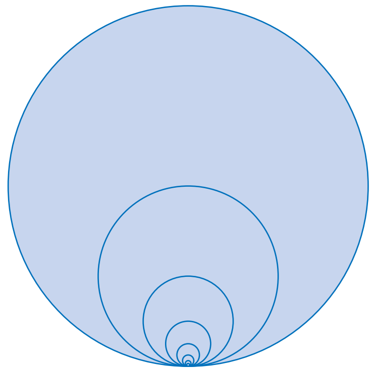
17. The third term of a geometric series is 24 and the fourth term is 36. Determine the sum of the first 10 terms. Express your answer as an exact fraction.

18. Three numbers, a, b, and c, form a geometric series so that $a + b + c = 35$ and $abc = 1000$. What are the values of a, b, and c?

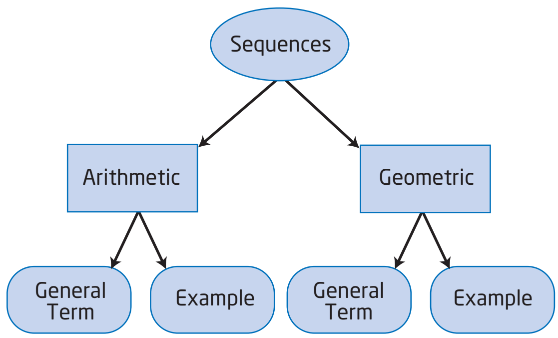
19. The sum of the first 7 terms of a geometric series is 89, and the sum of the first 8 terms is 104. What is the value of the eighth term?

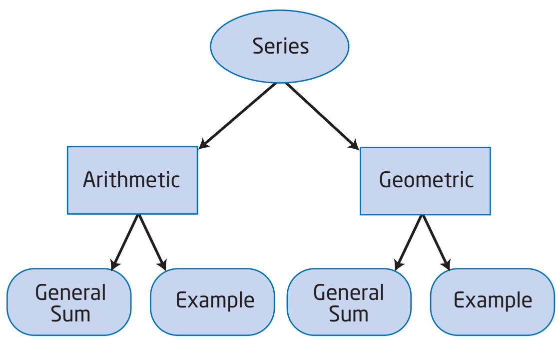
Create Connections

20. A fractal is created as follows: A circle is drawn with radius 8 cm. Another circle is drawn with half the radius of the previous circle. The new circle is tangent to the previous circle at point T as shown. Suppose this pattern continues through five steps. What is the sum of the areas of the circles? Express your answer as an exact fraction.



21. Copy the following flowcharts. In the appropriate segment of each chart, give a definition, a general term or sum, or an example, as required.





22. Tom learned that the monarch butterfly lays an average of 400 eggs. He decided to calculate the growth of the butterfly population from a single butterfly by using the logic that the first butterfly produced 400 butterflies. Each of those butterflies would produce 400 butterflies, and this pattern would continue. Tom wanted to estimate how many butterflies there would be in total in the fifth generation following this pattern. His calculation is shown below.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_5 = \frac{1(400^5 - 1)}{400 - 1}$$

$$S_5 \approx 2.566 \times 10^{10}$$

Tom calculated that there would be approximately 2.566×10^{10} monarch butterflies in the fifth generation.

a) What assumptions did Tom make in his calculations?

b) Do you agree with the method Tom used to arrive at the number of butterflies? Explain.

c) Would this be a reasonable estimate of the total number of butterflies in the fifth generation? Explain.

d) Explain a method you would use to calculate the number of butterflies in the fifth generation.

Did You Know?

According to the American Indian Butterfly Legend:

If anyone desires a wish to come true they must first capture a butterfly and whisper that wish to it.

Since a butterfly can make no sound, the butterfly cannot reveal the wish to anyone but the Great Spirit who hears and sees all.

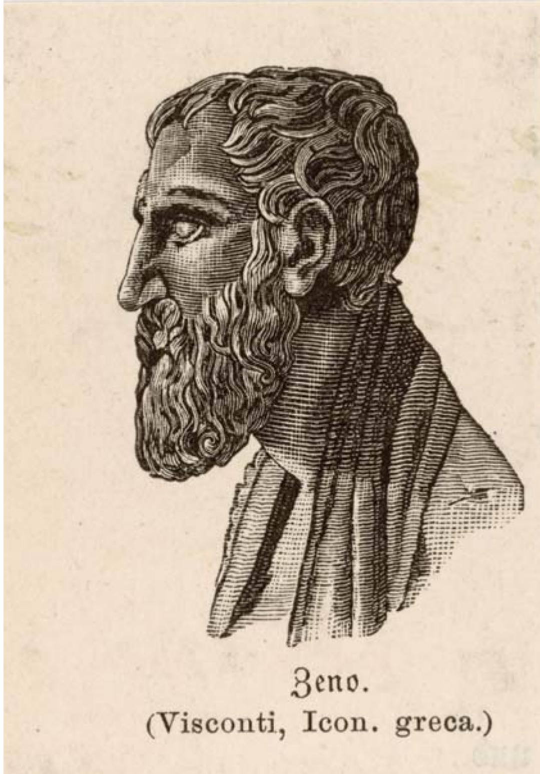
In gratitude for giving the beautiful butterfly its freedom, the Great Spirit always grants the wish.

So, according to legend, by making a wish and giving the butterfly its freedom, the wish will be taken to the heavens and be granted.

Project Corner — Oil Discovery

- The first oil well in Canada was discovered by James Miller Williams in 1858 near Oil Springs, Ontario. The oil was taken to Hamilton, Ontario, where it was refined into lamp oil. This well produced 37 barrels a day. By 1861 there were 400 wells in the area.
- In 1941, Alberta's population was approximately 800 000. By 1961, it was about 1.3 million.
- In February 1947, oil was struck in Leduc, Alberta. Leduc was the largest discovery in Canada in 33 years. By the end of 1947, 147 more wells were drilled in the Leduc-Woodbend oilfield.
- With these oil discoveries came accelerated population growth. In 1941, Leduc was inhabited by 871 people. By 1951, its population had grown to 1842.
- Leduc #1 was capped in 1974, after producing 300 000 barrels of oil and 9 million cubic metres of natural gas.

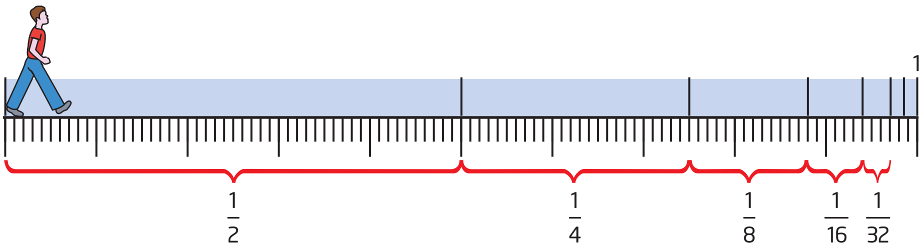
Infinite Geometric Series



Focus on...

- generalizing a rule for determining the sum of an infinite geometric series
- explaining why a geometric series is convergent or divergent
- solving a problem that involves a geometric sequence or series

In the fifth century B.C.E., the Greek philosopher Zeno of Elea posed four problems, now known as Zeno's paradoxes. These problems were intended to challenge some of the ideas that were held in his day. His paradox of motion states that a person standing in a room cannot walk to the wall. In order to do so, the person would first have to go half the distance, then half the remaining distance, and then half of what still remains. This process can always be continued and can never end.



Zeno's argument is that there is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on to infinity.

Where does the argument break down? Why?

Did You Know?

The word *paradox* comes from the Greek *para doxa*, meaning something contrary to opinion.

Investigate an Infinite Series

Materials

- square piece of paper
- ruler

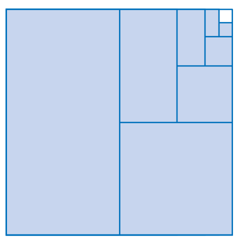
1. Start with a square piece of paper.

a) Draw a line dividing it in half.

b) Shade one of the halves.

c) In the unshaded half of the square, draw a line to divide it in half. Shade one of the halves.

d) Repeat part c) at least six more times.



2. Write a sequence of terms indicating the area of each newly shaded region as a fraction of the entire page. List the first five terms.

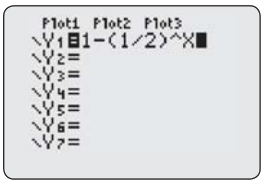
3. Predict the next two terms for the sequence.

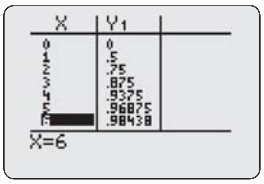
4. Is the sequence arithmetic, geometric, or neither? Justify your answer.
5. Write the rule for the nth term of the sequence.
6. Ignoring physical limitations, could this sequence continue indefinitely? In other words, would this be an infinite sequence? Explain your answer.
7. What conclusion can you make about the area of the square that would remain unshaded as the number of terms in the sequence approaches infinity?

Reflect and Respond

8. Using a graphing calculator, input the function $y = \left(\frac{1}{2}\right)^x$.
   a) Using the table of values from the calculator, what happens to the value of $y = \left(\frac{1}{2}\right)^x$ as x gets larger and larger?
   b) Can the value of $\left(\frac{1}{2}\right)^x$ ever equal zero?
9. The geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ can be written as $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^x$.
   You can use the general formula to determine the sum of the series.
   $$S_x = \frac{t_1(1 - r^x)}{(1 - r)}$$
   $$S_x = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^x\right)}{1 - \frac{1}{2}}$$
   $$S_x = 1 - \left(\frac{1}{2}\right)^x$$
   For values of $r < 1$, the general formula $S_x = \frac{t_1(r^x - 1)}{(r - 1)}$ can be written for convenience as $S_x = \frac{t_1(1 - r^x)}{(1 - r)}$. Why do you think this is true?

   Enter the function into your calculator and use the table feature to find the sum, S_x, as x gets larger.



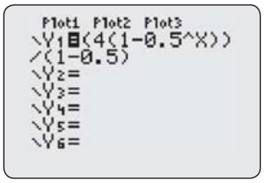


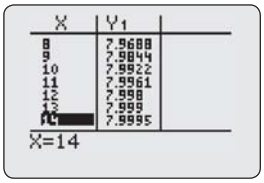
   a) What happens to the sum, S_x, as x gets larger?
   b) Will the sum increase without limit? Explain your reasoning.
10. a) As the value of x gets very large, what value can you assume that r^x becomes close to?
   b) Use your answer from part a) to modify the formula for the sum of a geometric series to determine the sum of an infinite geometric series.
   c) Use your formula from part b) to determine the sum of the infinite geometric series $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$.

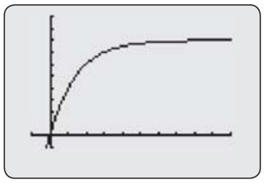
Link the Ideas

Convergent Series

Consider the series $4 + 2 + 1 + 0.5 + 0.25 + \cdots$







$S_5 = 7.75$
$S_7 = 7.9375$
$S_9 = 7.9844$
$S_{11} = 7.9961$
$S_{13} = 7.999$
$S_{15} = 7.9998$
$S_{17} = 7.9999$

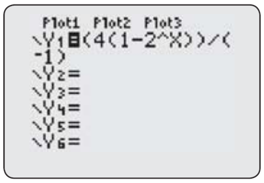
As the number of terms increases, the sequence of partial sums approaches a fixed value of 8. Therefore, the sum of this series is 8. This series is said to be a **convergent series**.

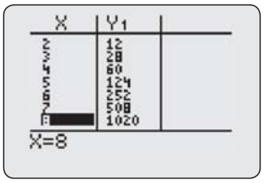
convergent series
- a series with an infinite number of terms, in which the sequence of partial sums approaches a fixed value
- for example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

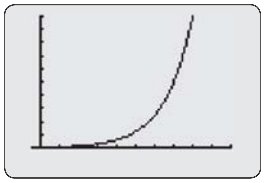
Divergent Series

Consider the series $4 + 8 + 16 + 32 + \cdots$

$S_1 = 4$
$S_2 = 12$
$S_3 = 28$
$S_4 = 60$
$S_5 = 124$







As the number of terms increases, the sum of the series continues to grow. The sequence of partial sums does not approach a fixed value. Therefore, the sum of this series cannot be calculated. This series is said to be a **divergent series**.

divergent series
- a series with an infinite number of terms, in which the sequence of partial sums does not approach a fixed value
- for example, $2 + 4 + 8 + 16 + \cdots$

Infinite Geometric Series

The formula for the sum of a geometric series is

$$S_n = \frac{t_1(1 - r^n)}{1 - r}.$$

As n gets very large, the value of the r^n approaches 0, for values of r between -1 and 1.

So, as n gets large, the partial sum S_n approaches $\frac{t_1}{1 - r}$.

Therefore, the sum of an infinite geometric series is

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1.$$

The *sum* of an infinite geometric series, where $-1 < r < 1$, can be determined using the formula

$$\boldsymbol{S_\infty = \frac{t_1}{1 - r}}$$

where t_1 is the first term of the series
r is the common ratio
S_∞ represents the sum of an infinite number of terms

Applying the formula to the series $4 + 2 + 1 + 0.5 + 0.25 + \cdots$

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$
$$S_\infty = \frac{4}{1 - 0.5}$$
$$S_\infty = \frac{4}{0.5}$$
$$S_\infty = 8$$

Example 1
Sum of an Infinite Geometric Series

Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

a) $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

b) $2 - 4 + 8 - \cdots$

Solution

a) $t_1 = 1$, $r = -\frac{1}{3}$

Since $-1 < r < 1$, the series is convergent.
Use the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$
$$S_\infty = \frac{1}{1 - \left(-\frac{1}{3}\right)}$$
$$S_\infty = \frac{1}{\frac{4}{3}}$$
$$S_\infty = (1)\left(\frac{3}{4}\right)$$
$$S_\infty = \frac{3}{4}$$

b) $t_1 = 2$, $r = -2$

Since $r < -1$, the series is divergent and has no sum.

Your Turn

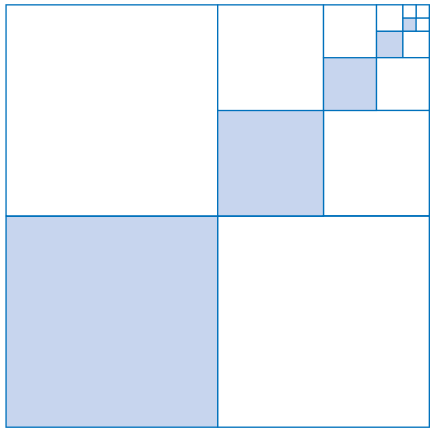
Determine whether each infinite geometric series converges or diverges. Calculate the sum, if it exists.

a) $1 + \frac{1}{5} + \frac{1}{25} + \cdots$

b) $4 + 8 + 16 + \cdots$

Example 2

Apply the Sum of an Infinite Geometric Series



Assume that each shaded square represents $\frac{1}{4}$ of the area of the larger square bordering two of its adjacent sides and that the shading continues indefinitely in the indicated manner.

a) Write the series of terms that would represent this situation.

b) How much of the total area of the largest square is shaded?

Solution

a) The sequence of shaded regions generates an infinite geometric sequence. The series of terms that represents this situation is
$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$

b) To determine the total area shaded, you need to determine the sum of all the shaded regions within the largest square.

For this series,

First term $t_1 = \frac{1}{4}$

Common ratio $r = \frac{1}{4}$

Use the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$

$$S_\infty = \frac{\frac{1}{4}}{1 - \frac{1}{4}}$$

$$S_\infty = \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$S_\infty = \left(\frac{1}{4}\right)\left(\frac{4}{3}\right)$$

$$S_\infty = \frac{1}{3}$$

A total area of $\frac{1}{3}$ of the largest square is shaded.

Your Turn

You can express $0.\overline{584}$ as an infinite geometric series.

$$\begin{aligned} 0.\overline{584} &= 0.584\ 584\ 584\ldots \\ &= 0.584 + 0.000\ 584 + 0.000\ 000\ 584 + \cdots \end{aligned}$$

Determine the sum of the series.

Key Ideas

- An infinite geometric series is a geometric series that has an infinite number of terms; that is, the series has no last term.
- An infinite series is said to be convergent if its sequence of partial sums approaches a finite number. This number is the sum of the infinite series. An infinite series that is not convergent is said to be divergent.
- An infinite geometric series has a sum when $-1 < r < 1$ and the sum is given by $S_\infty = \frac{t_1}{1 - r}$.

Check Your Understanding

Practise

1. State whether each infinite geometric series is convergent or divergent.

a) $t_1 = -3$, $r = 4$

b) $t_1 = 4$, $r = -\frac{1}{4}$

c) $125 + 25 + 5 + \cdots$

d) $(-2) + (-4) + (-8) + \cdots$

e) $\frac{243}{3125} - \frac{81}{625} + \frac{27}{25} - \frac{9}{5} + \cdots$

2. Determine the sum of each infinite geometric series, if it exists.

a) $t_1 = 8$, $r = -\frac{1}{4}$

b) $t_1 = 3$, $r = \frac{4}{3}$

c) $t_1 = 5$, $r = 1$

d) $1 + 0.5 + 0.25 + \cdots$

e) $4 - \frac{12}{5} + \frac{36}{25} - \frac{108}{125} + \cdots$

3. Express each of the following as an infinite geometric series. Determine the sum of the series.

a) $0.\overline{87}$

b) $0.4\overline{37}$

4. Does $0.999... = 1$? Support your answer.

5. What is the sum of each infinite geometric series?

a) $5 + 5\left(\frac{2}{3}\right) + 5\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right)^3 + \cdots$

b) $1 + \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^3 + \cdots$

c) $7 + 7\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right)^3 + \cdots$

Apply

6. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. What is the value of the first term? Write the first three terms of the series.

7. The first term of an infinite geometric series is -8, and its sum is $-\frac{40}{3}$. What is the common ratio? Write the first four terms of the series.

8. In its first month, an oil well near Virden, Manitoba produced 24 000 barrels of crude. Every month after that, it produced 94% of the previous month's production.

a) If this trend continued, what would be the lifetime production of this well?

b) What assumption are you making? Is your assumption reasonable?

9. The infinite series given by $1 + 3x + 9x^2 + 27x^3 + \cdots$ has a sum of 4. What is the value of x? List the first four terms of the series.

10. The sum of an infinite series is twice its first term. Determine the value of the common ratio.

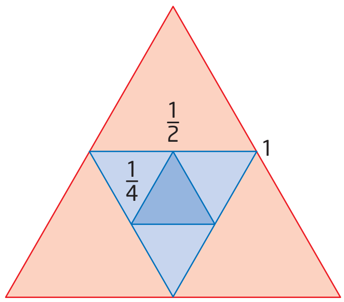
11. Each of the following represents an infinite geometric series. For what values of x will each series be convergent?

    a) $5 + 5x + 5x^2 + 5x^3 + \cdots$

    b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots$

    c) $2 + 4x + 8x^2 + 16x^3 + \cdots$

12. Each side of an equilateral triangle has length of 1 cm. The midpoints of the sides are joined to form an inscribed equilateral triangle. Then, the midpoints of the sides of that triangle are joined to form another triangle. If this process continues forever, what is the sum of the perimeters of the triangles?



13. The length of the initial swing of a pendulum is 50 cm. Each successive swing is 0.8 times the length of the previous swing. If this process continues forever, how far will the pendulum swing?

14. Andrew uses the formula for the sum of an infinite geometric series to evaluate $1 + 1.1 + 1.21 + 1.331 + \cdots$. He calculates the sum of the series to be 10. Is Andrew's answer reasonable? Explain.

15. A ball is dropped from a height of 16 m. The ball rebounds to one half of its previous height each time it bounces. If the ball keeps bouncing, what is the total vertical distance the ball travels?

16. A pile driver pounds a metal post into the ground. With the first impact, the post moves 30 cm; with the second impact it moves 27 cm. Predict the total distance that the post will be driven into the ground if

    a) the distances form a geometric sequence and the post is pounded 8 times

    b) the distances form a geometric sequence and the post is pounded indefinitely

17. Dominique and Rita are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \cdots$. Dominique says that the sum of the series is $-\frac{1}{7}$. Rita says that the series is divergent and has no sum.

    a) Who is correct?

    b) Explain your reasoning.

18. A hot air balloon rises 25 m in its first minute of flight. Suppose that in each succeeding minute the balloon rises only 80% as high as in the previous minute. What would be the balloon's maximum altitude?

Hot air balloon rising over Calgary.

Extend

19. A square piece of paper with a side length of 24 cm is cut into four small squares, each with side lengths of 12 cm. Three of these squares are placed side by side. The remaining square is cut into four smaller squares, each with side lengths of 6 cm. Three of these squares are placed side by side with the bigger squares. The fourth square is cut into four smaller squares and three of these squares are placed side by side with the bigger squares. Suppose this process continues indefinitely. What is the length of the arrangement of squares?

20. The sum of the series
$0.98 + 0.98^2 + 0.98^3 + \cdots + 0.98^n = 49.$
The sum of the series
$0.02 + 0.0004 + 0.000\ 008 + \cdots = \frac{1}{49}.$
The common ratio in the first series is 0.98 and the common ratio in the second series is 0.02. The sum of these ratios is equal to 1. Suppose that $\frac{1}{z} = x + x^2 + x^3 + \cdots,$ where z is an integer and $x = \frac{1}{z + 1}.$

a) Create another pair of series that would follow this pattern, where the sum of the common ratios of the two series is 1.

b) Determine the sum of each series using the formula for the sum of an infinite series.

Create Connections

21. Under what circumstances will an infinite geometric series converge?

22. The first two terms of a series are 1 and $\frac{1}{4}$. Determine a formula for the sum of n terms if the series is

a) an arithmetic series

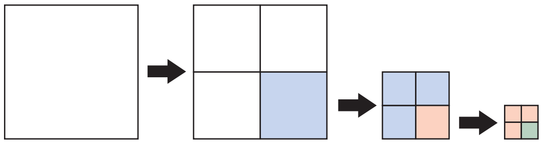
b) a geometric series

c) an infinite geometric series

23. MINI LAB Work in a group of three.

Step 1 Begin with a large sheet of grid paper and draw a square. Assume that the area of this square is 1.

Step 2 Cut the square into 4 equal parts. Distribute one part to each member of your group. Cut the remaining part into 4 equal parts. Again distribute one part to each group member. Subdivide the remaining part into 4 equal parts. Suppose you could continue this pattern indefinitely.



Step 3 Write a sequence for the fraction of the original square that each student received at each stage.

n	1	2	3	4
Fraction of Paper				

Step 4 Write the total area of paper each student has as a series of partial sums. What do you expect the sum to be?

Project Corner — Petroleum

- The Athabasca Oil Sands have estimated oil reserves in excess of that of the rest of the world. These reserves are estimated to be 1.6 trillion barrels.
- Canada is the seventh largest oil producing country in the world. In 2008, Canada produced an average of 438 000 m^3 per day of crude oil, crude bitumen, and natural gas.
- As Alberta's reserves of light crude oil began to deplete, so did production. By 1997, Alberta's light crude oil production totalled 37.3 million cubic metres. This production has continued to decline each year since, falling to just over half of its 1990 total at 21.7 million cubic metres in 2005.

Chapter 1 Review

1.1 Arithmetic Sequences, pages 6–21

1. Determine whether each of the following sequences is arithmetic. If it is arithmetic, state the common difference.

a) 36, 40, 44, 48, …

b) −35, −40, −45, −50, …

c) 1, 2, 4, 8, …

d) 8.3, 4.3, 0.3, −3, −3.7, …

2. Match the equation for the *n*th term of an arithmetic sequence to the correct sequence.

a) 18, 30, 42, 54, 66, …	**A** $t_n = 3n + 1$
b) 7, 12, 17, 22, …	**B** $t_n = -4(n + 1)$
c) 2, 4, 6, 8, …	**C** $t_n = 12n + 6$
d) −8, −12, −16, −20, …	**D** $t_n = 5n + 2$
e) 4, 7, 10, 13, …	**E** $t_n = 2n$

3. Consider the sequence 7, 14, 21, 28, … . Determine whether each of the following numbers is a term of this sequence. Justify your answer. If the number is a term of the sequence, determine the value of *n* for that term.

a) 98 **b)** 110

c) 378 **d)** 575

4. Two sequences are given:
Sequence 1 is 2, 9, 16, 23, …
Sequence 2 is 4, 10, 16, 22, …

a) Which of the following statements is correct?

A t_{17} is greater in sequence 1.

B t_{17} is greater in sequence 2.

C t_{17} is equal in both sequences.

b) On a grid, sketch a graph of each sequence. Does the graph support your answer in part a)? Explain.

5. Determine the tenth term of the arithmetic sequence in which the first term is 5 and the fourth term is 17.

6. The Gardiner Dam, located 100 km south of Saskatoon, Saskatchewan, is the largest earth-filled dam in the world. Upon its opening in 1967, engineers discovered that the pressure from Lake Diefenbaker had moved the clay-based structure 200 cm downstream. Since then, the dam has been moving at a rate of 2 cm per year. Determine the distance the dam will have moved downstream by the year 2020.

1.2 Arithmetic Series, pages 22–31

7. Determine the indicated sum for each of the following arithmetic series

a) $6 + 9 + 12 + \cdots (S_{10})$

b) $4.5 + 8 + 11.5 + \cdots (S_{12})$

c) $6 + 3 + 0 + \cdots (S_{10})$

d) $60 + 70 + 80 + \cdots (S_{20})$

8. The sum of the first 12 terms of an arithmetic series is 186, and the 20th term is 83. What is the sum of the first 40 terms?

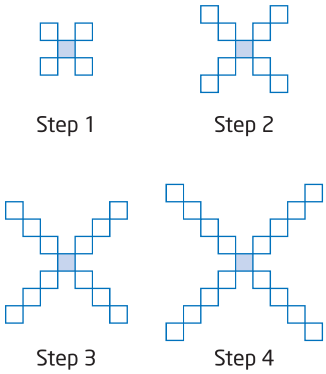
9. You have taken a job that requires being in contact with all the people in your neighbourhood. On the first day, you are able to contact only one person. On the second day, you contact two more people than you did on the first day. On day three, you contact two more people than you did on the previous day. Assume that the pattern continues.

a) How many people would you contact on the 15th day?

b) Determine the total number of people you would have been in contact with by the end of the 15th day.

c) How many days would you need to contact the 625 people in your neighbourhood?

10. A new set of designs is created by the addition of squares to the previous pattern.



a) Determine the total number of squares in the 15th step of this design.

b) Determine the total number of squares required to build all 15 steps.

11. A concert hall has 10 seats in the first row. The second row has 12 seats. If each row has 2 seats more than the row before it and there are 30 rows of seats, how many seats are in the entire concert hall?

1.3 Geometric Sequences, pages 32–45

12. Determine whether each of the following sequences is geometric. If it is geometric, determine the common ratio, r, the first term, t_1, and the general term of the sequence.

a) 3, 6, 10, 15, …

b) $1, -2, 4, -8, \ldots$

c) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

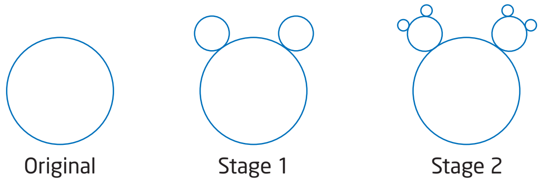
d) $\frac{16}{9}, -\frac{3}{4}, 1, \ldots$

13. A culture initially has 5000 bacteria, and the number increases by 8% every hour.

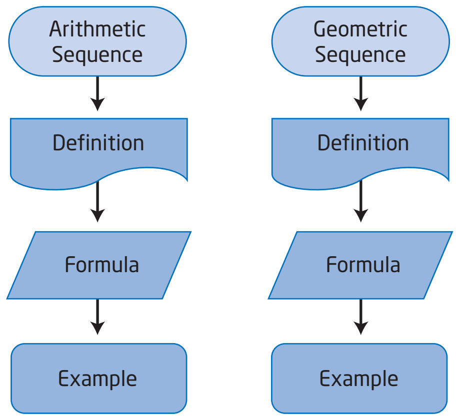
a) How many bacteria are present at the end of 5 h?

b) Determine a formula for the number of bacteria present after n hours.

14. In the Mickey Mouse fractal shown below, the original diagram has a radius of 81 cm. Each successive circle has a radius $\frac{1}{3}$ of the previous radius. What is the circumference of the smallest circle in the 4th stage?



15. Use the following flowcharts to describe what you know about arithmetic and geometric sequences.



1.4 Geometric Series, pages 46–57

16. Decide whether each of the following statements relates to an arithmetic series or a geometric series.

a) A sum of terms in which the difference between consecutive terms is constant.

b) A sum of terms in which the ratio of consecutive terms is constant.

c) $S_n = \frac{t_1(r^n - 1)}{r - 1}, r \neq 1$

d) $S_n = \frac{n[2t_1 + (n - 1)d]}{2}$

e) $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \cdots$

f) $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{2}{27} + \cdots$

17. Determine the sum indicated for each of the following geometric series.

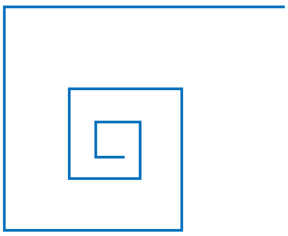
a) $6 + 9 + 13.5 + \cdots (S_{10})$

b) $18 + 9 + 4.5 + \cdots (S_{12})$

c) $6000 + 600 + 60 + \cdots (S_{20})$

d) $80 + 20 + 5 + \cdots (S_9)$

18. A student programs a computer to draw a series of straight lines with each line beginning at the end of the previous line and at right angles to it. The first line is 4 mm long. Each subsequent line is 25% longer than the previous one, so that a spiral shape is formed as shown.



a) What is the length, in millimetres, of the eighth straight line drawn by the program? Express your answer to the nearest tenth of a millimetre.

b) Determine the total length of the spiral, in metres, when 20 straight lines have been drawn. Express your answer to the nearest hundredth of a metre.

1.5 Infinite Geometric Series, pages 58–65

19. Determine the sum of each of the following infinite geometric series.

a) $5 + 5\left(\frac{2}{3}\right) + 5\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right)^3 + \cdots$

b) $1 + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^3 + \cdots$

20. For each of the following series, state whether it is convergent or divergent. For those that are convergent, determine the sum.

a) $8 + 4 + 2 + 1 + \cdots$

b) $8 + 12 + 27 + 40.5 + \cdots$

c) $-42 + 21 - 10.5 + 5.25 - \cdots$

d) $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots$

21. Given the infinite geometric series: $7 - 2.8 + 1.12 - 0.448 + \cdots$

a) What is the common ratio, r?

b) Determine S_1, S_2, S_3, S_4, and S_5.

c) What is the particular value that the sums are approaching?

d) What is the sum of the series?

22. Draw four squares adjacent to each other. The first square has a side length of 1 unit, the second has a side length of $\frac{1}{2}$ unit, the third has a side length of $\frac{1}{4}$ unit, and the fourth has a side length of $\frac{1}{8}$ unit.

a) Calculate the area of each square. Do the areas form a geometric sequence? Justify your answer.

b) What is the total area of the four squares?

c) If the process of adding squares with half the side length of the previous square continued indefinitely, what would the total area of all the squares be?

23. a) Copy and complete each of the following statements.

- A series is geometric if there is a common ratio r such that ______.
- An infinite geometric series converges if ______.
- An infinite geometric series diverges if ______.

b) Give two examples of convergent infinite geometric series one with positive common ratio and one with negative common ratio. Determine the sum of each of your series.

Chapter 1 Practice Test

Multiple Choice

For # 1 to #5, choose the best answer.

1. What are the missing terms of the arithmetic sequence ■, 3, 9, ■, ■?

A 1, 27, 81 **B** 9, 3, 9

C −6, 12, 17 **D** −3, 15, 21

2. Marc has set up in his father's grocery store a display of cans as shown in the diagram. The top row (Row 1) has 1 can and each successive row has 3 more cans than the previous row. Which expression would represent the number of cans in row n?

A $S_n = 3n + 1$ **B** $t_n = 3n - 2$

C $t_n = 3n + 2$ **D** $S_n = 3n - 3$

3. What is the sum of the first five terms of the geometric series 16 807 − 2401 + 343 − ⋯?

A 19 607 **B** 14 707

C 16 807.29 **D** 14 706.25

4. The numbers represented by a, b, and c are the first three terms of an arithmetic sequence. The number c, when expressed in terms of a and b, would be represented by

A $a + b$ **B** $2b - a$

C $a + (n - 1)b$ **D** $2a + b$

5. The 20th term of a geometric sequence is 524 288 and the 14th term is 8192. The value of the third term could be

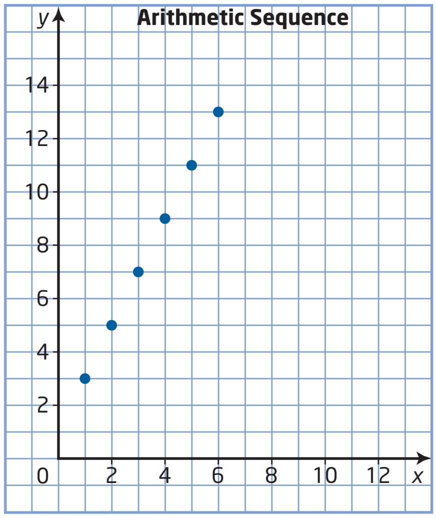
A 4 only **B** 8 only

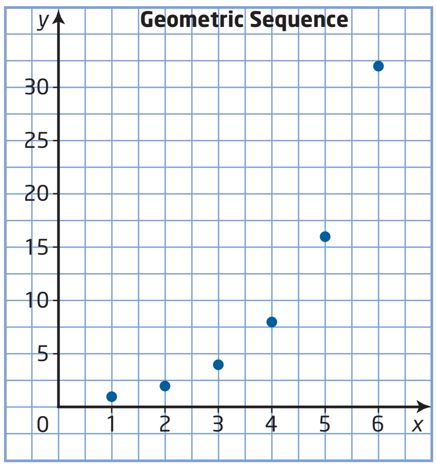
C +4 or −4 **D** +8 or −8

Short Answer

6. A set of hemispherical bowls are made so they can be nested for easy storage. The largest bowl has a radius of 30 cm and each successive bowl has a radius 90% of the preceding one. What is the radius of the tenth bowl?

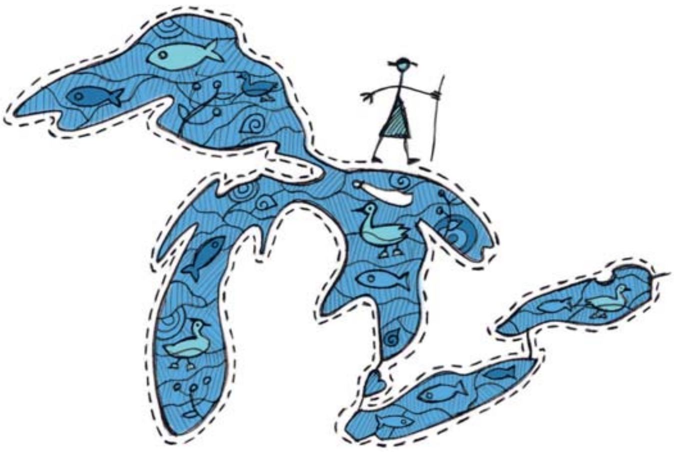
7. Use the following graphs to compare and contrast an arithmetic and a geometric sequence.





8. If 3, A, 27 is an arithmetic sequence and 3, B, 27 is a geometric sequence where $B > 0$, then what are the values of A and B?

9. Josephine Mandamin, an Anishinabe elder from Thunder Bay, Ontario, set out to walk around the Great Lakes to raise awareness about the quality of water in the lakes. In six years, she walked 17 000 km. If Josephine increased the number of kilometres walked per week by 2% every week, how many kilometres did she walk in the first week?



10. Consider the sequence 5, ■, ■, ■, ■, 160.

a) Assume the sequence is arithmetic. Determine the unknown terms of the sequence.

b) What is the general term of the arithmetic sequence?

c) Assume the sequence is geometric. Determine the unknown terms of the sequence.

d) What is the general term of the geometric sequence?

Extended Response

11. Scientists have been measuring the continental drift between Europe and North America for about 25 years. The data collected show that the continents are moving apart at a steady rate of about 17 mm per year.

a) According to the Pangaea theory, Europe and North America were connected at one time. Assuming this theory is correct, write an arithmetic sequence that describes how far apart the continents were at the end of each of the first five years after separation.

b) Determine the general term that describes the arithmetic sequence.

c) Approximately how many years did it take to separate to the current distance of 6000 km? Express your answer to the nearest million years.

d) What assumptions did you make in part c)?

12. Photodynamic therapy is used in patients with certain types of disease. A doctor injects a patient with a drug that is attracted to the diseased cells. The diseased cells are then exposed to red light from a laser. This procedure targets and destroys diseased cells while limiting damage to surrounding healthy tissue. The drug remains in the normal cells of the body and must be bleached out by exposure to the sun. A patient must be exposed to the sun for 30 s on the first day, and then increase the exposure by 30 s every day until a total of 30 min is reached.

a) Write the first five terms of the sequence of sun exposure times.

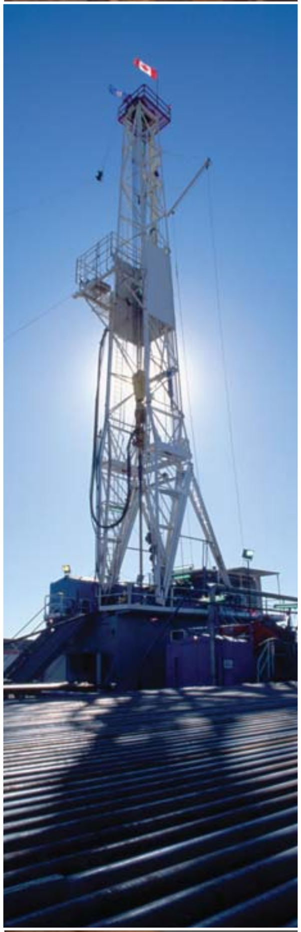
b) Is the sequence arithmetic or geometric?

c) How many days are required to reach the goal of 30 min of exposure to the sun?

d) What is the total number of minutes of sun exposure when a patient reaches the 30 min goal?

Unit 1 Project

Canada's Natural Resources



Canada is the source of more than 60 mineral commodities, including metals, non-metals, structural materials, and mineral fuels.

Quarrying and mining are among the oldest industries in Canada. In 1672, coal was discovered on Cape Breton Island.

In the 1850s, gold discoveries in British Columbia, oil finds in Ontario, and increased production of Cape Breton coal marked a turning point in Canadian mineral history.

In 1896, gold was found in the Klondike District of what became Yukon Territory, giving rise to one of the world's most spectacular gold rushes.

In the late 1800s, large deposits of coal and oil sands were evident in part of the North-West Territories that later became Alberta.

In the post-war era there were many major mineral discoveries: deposits of nickel in Manitoba; zinc-lead, copper, and molybdenum in British Columbia; and base metals and asbestos in Québec, Ontario, Manitoba, Newfoundland, Yukon Territory, and British Columbia.

The discovery of the famous Leduc oil field in Alberta in 1947 was followed by a great expansion of Canada's petroleum industry.

In the late 1940s and early 1950s, uranium was discovered in Saskatchewan and Ontario. In fact, Canada is now the world's largest uranium producer.

Canada's first diamond-mining operation began production in October 1998 at the Ekati mine in Lac de Gras, Northwest Territories, followed by the Diavik mine in 2002.

Chapter 1 Task

Choose a natural resource that you would like to research. You may wish to look at some of the information presented in the Project Corner boxes throughout Chapter 1 for ideas. Research your chosen resource.

- List interesting facts about your chosen resource, including what it is, how it is produced, where it is exported, how much is exported, and so on.
- Look for data that would support using a sequence or series in discussing or describing your resource. List the terms for the sequence or series you include.
- Use the information you have gathered in a sequence or series to predict possible trends in the use or production of the resource over a ten-year period.
- Describe any effects the production of the natural resource has on the community.