Sequences and Series

KNOW	DO	UNDERSTAND
Sequences are functioning whose domain is discrete. Can identify if a sequence is geometric.	Can use sigma notation to describe series. Can build the equation for a geometric sequence. Can determine the sum of a geometric series.	None yet

Vocab & Notation

- Sequence, $(a_n)_{n\in\mathbb{N}}$
- Index
- Recursion
- Series
- Sigma Notation, $\sum k$
- Geometric
- Common Ratio

Definition: A **sequence** is an ordered collection of objects. Like a set, but order matters. It is denoted as

$$(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots)$$

The position of an element of a sequence is called the **index**, and there is a relationship between the index and the **term.**

$$s: \mathbb{N} \to \mathbb{R}$$
$$s: n \mapsto a_n$$

Examples:

$(2, 4, 6, 8, \dots)$	A005843
$(2, 4, 8, 16, \dots)$	A000079
$(2, 3, 5, 7, 11, \dots)$	A000040
$(2, 3, 5, 8, 13, \dots)$	A000045

There is an online encyclopedia of integer sequences called OEIS and found at “oeis.org”. The sequence number is shown on the side.

Oftentimes it is helpful to define a sequence starting with index 0 instead of 1. Really any number can be used to start the index but 0 and 1 are the main choices.

Definition: A **geometric sequence** is a sequence generated by multiplying the previous term by a fixed value.

$$a_{k+1} = a_k \cdot r$$

Definition: The **common ratio** is the ratio of consecutive terms in a geometric sequence.

$$\frac{a_{k+1}}{a_k} = r$$

Let's look and see how a geometric sequence is built by letting $a_1 = A$

Example: Determine sequences of the form $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ for the following patterns:

$(2, 6, 18, \cdots)$

$(100, -50, 25, \cdots)$

$\left(\frac{8}{9}, \frac{2}{9}, \frac{1}{18}, \cdots\right)$

$\left(-9, -25, -\frac{625}{9}, \cdots\right)$

Definition: A **series** is a sum of a sequence, either finite or infinite.

Example: Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_k = 4 \cdot \frac{(-1)^{k+1}}{2k-1}$, then we have that the sequence is:

And we can add the first 5 terms or add all of them as

$$S_5 = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} = \frac{1052}{315} = 3.3396 \ldots$$

$$S_\infty = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots =$$

We'll get to the last sum in a second, but there is a better way to express the summation using **sigma notation**

$$S_n = \sum_{k=1}^{n} a_k$$

So the sums we were looking at can be written as

$$S_5 = \sum_{k=1}^{5} \frac{4 \cdot (-1)^{k+1}}{2k-1} \qquad S = \sum_{k=1}^{\infty} \frac{4 \cdot (-1)^{k+1}}{2k-1}$$

Typing this into Desmos we get the sigma notation by typing "sum".

Practice: Write out the terms of the following sums and compute the sum

$$\sum_{k=1}^{5} 3k \qquad \sum_{n=1}^{4} \frac{3n+2}{n+1} \qquad \sum_{i=0}^{3} 2^i$$

Practice: Write the following series in sigma notation:

$$8 + 24 + 72 + 216 + 648$$

$$\frac{2}{9} - \frac{2}{3} + 2 - 6 + 18 - 54$$

Now we would like to consider the geometric series, that is

$$S_n = \sum_{k=1}^{n} a_1 \cdot r^{k-1}$$

Oftentimes it is helpful to try and add or subtract copies of the summation to reduce it to something simpler.

Example: Determine the following sums

$$\sum_{k=1}^{10} 4 \cdot 3^{k-1}$$

$$\sum_{k=0}^{5} 3 \cdot \left(\frac{1}{2}\right)^{k}$$

Practice: Determine the following sums

$$\sum_{k=1}^{7} 8 \cdot \left(-\frac{2}{3}\right)^{k-1}$$

$$\sum_{k=0}^{6} \left(\frac{9}{4}\right)^{k}$$

Since we know the finite sum, we can consider what S_∞ would be:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Example: Determine the following sums

$$\sum_{k=1}^{\infty} 10 \cdot \left(-\frac{2}{5}\right)^{k-1}$$

$$\sum_{k=0}^{\infty} 7 \cdot \left(-\frac{3}{4}\right)^{k}$$

$$\sum_{k=2}^{\infty} 100 \cdot \left(\frac{1}{2}\right)^{k}$$

$$\sum_{k=0}^{\infty} 12 \cdot (1.01)^{k}$$

Practice Problems: Handout sections I, II, III, V(a,c,d)
Sigma Notation Module:
@ blogs.ubc.ca/infiniteseriesmodule/units/unit-1/sigma-notation/
Ryerson pdf: 1.3 # 1-7, 23
1.4 # 1-8, 16-19
1.5 # 1-7, 20