Sequences and Series

KNOW	DO	UNDERSTAND
Sequences are functioning whose domain is discrete. Can identify if a sequence is geometric.	Can use sigma notation to describe series. Can build the equation for a geometric sequence. Can determine the sum of a geometric series.	None yet

Vocab & Notation

- Sequence, $(a_n)_{n\in\mathbb{N}}$
- Index
- Recursion
- Series
- Sigma Notation, $\sum k$
- Geometric
- Common Ratio

Definition: A **sequence** is an ordered collection of objects. Like a set, but order matters. It is denoted as

$$(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots)$$

index term

The position of an element of a sequence is called the **index**, and there is a relationship between the index and the **term.**

$$s: \mathbb{N} \to \mathbb{R}$$
$$s: n \mapsto a_n$$

Examples:

$(2, 4, 6, 8, \dots)$ 10, 12, $(2n)_{n\in\mathbb{N}}$	$a_{n+1} = a_n + 2$ (recursive), $a_1 = 2$ $a_n = 2n$ (closed)	A005843
$2^1\ 2^2\ 2^3\ 2^4$ $(2, 4, 8, 16, \dots)$ $(2^n)_{n=1}^{\infty}$	$a_{n+1} = 2\cdot a_n$, $a_1 = 2$ $a_n = 2^n$	A000079
$(2, 3, 5, 7, 11, \dots)$ 13, 17, 19, 23, 29, 31	$a_n = n^{th}$ prime $\cancel{1}, 2, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, \cancel{15}, \cancel{16}, 17, \cancel{18}, 19$	A000040
$(2, 3, 5, 8, 13, \dots)$ 21, 34, 55 Fibonacci	$a_{n+1} = a_n + a_{n-1}$, $a_1 = 2$ and $a_2 = 3$ $\frac{1+\sqrt{5}}{2}$ $a_n \propto \varphi^n + \bar{\varphi}^n$ φ = golden ratio	A000045

There is an online encyclopedia of integer sequences called OEIS and found at "oeis.org". The sequence number is shown on the side.

Oftentimes it is helpful to define a sequence starting with index 0 instead of 1. Really any number can be used to start the index but 0 and 1 are the main choices.

$$(2n)_{n=1}^{\infty} = (2m)_{m=1}^{\infty} = (2(k+1))_{k=0}^{\infty} = (2, 4, 6, \dots)$$

let $k = m-1 \Rightarrow m = k+1$
if $m = 1 \to k = 0$

Definition: A **geometric sequence** is a sequence generated by multiplying the previous term by a fixed value.

$$a_{k+1} = a_k \cdot r$$

Definition: The **common ratio** is the ratio of consecutive terms in a geometric sequence.

true for every consecutive pair →

$$\frac{a_{k+1}}{a_k} = r$$

Let's look and see how a geometric sequence is built by letting $a_1 = A$

$$(a_n)_{n\in\mathbb{N}} = (A, Ar, Ar^2, Ar^3, \dots, Ar^{k-1}, \dots) = (Ar^{n-1})_{n\in\mathbb{N}}$$

$$= (a_1, a_2, a_3, a_4, \dots, a_k, \dots)$$

Example: Determine sequences of the form $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ for the following patterns:

$(2, 6, 18, \cdots)$

$\frac{6}{2} = \frac{18}{6} = 3$

$$(2 \cdot 3^{n-1})_{n=1}^{\infty} = (2 \cdot 3^k)_{k=0}^{\infty}$$

let $k = n-1$

if $n=1 \to k=0$

$(100, -50, 25, \cdots)$

$$\left(100 \cdot \left(-\tfrac{1}{2}\right)^{k-1}\right)_{k=1}^{\infty} = \left(100 \cdot \left(\tfrac{1}{-2}\right)^n\right)_{n=0}^{\infty}$$

$$= \left(\frac{100}{(-2)^n}\right)_{n=0}^{\infty}$$

$$= \left(100(-2)^{-n}\right)_{n=0}^{\infty}$$

$\left(\frac{8}{9}, \frac{2}{9}, \frac{1}{18}, \cdots\right)$

$r = \frac{1}{4}$

$$\left(\frac{8}{9} \cdot \left(\frac{1}{4}\right)^{n-1}\right)_{n=1}^{\infty} = \left(\frac{8}{9} \cdot \frac{1}{4^k}\right)_{k=0}^{\infty}$$

$\left(-9, -25, -\frac{625}{9}, \cdots\right)$

$r = \frac{25}{9}$

$$\left(-9\left(\frac{25}{9}\right)^k\right)_{k=0}^{\infty}$$

Definition: A **series** is a sum of a sequence, either finite or infinite.

$$(A \cdot r^n)_{n=0}^{\infty}$$

Example: Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_k = 4 \cdot \frac{(-1)^{k+1}}{2k-1}$, then we have that the sequence is:

$$(a_n)_{n=1}^{\infty} = \left(\frac{4}{1}, -\frac{4}{3}, \frac{4}{5}, -\frac{4}{7}, \frac{4}{9}, \dots\right)$$

$-\frac{7}{9}$ vs $-\frac{1}{3}$

NOT geometric

as $n \to \infty$ $a_n \to 0$

$1+1+1+1+\dots = \infty$

And we can add the first 5 terms or add all of them as

$$S_5 = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} = \frac{1052}{315} = 3.3396\ldots$$

$$S_\infty = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots =$$

We'll get to the last sum in a second, but there is a better way to express the summation using **sigma notation**

end @ k=n

sigma

$$(a_k)_{k=1}^{n} \qquad S_n = \sum_{k=1}^{n} a_k \qquad S_n = \sum_{k=1}^{n} a_k \leftarrow \text{index} \qquad = \sum_{i=1}^{n} a_i$$

start k=1

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

So the sums we were looking at can be written as

$$S_5 = \sum_{k=1}^{5} \frac{4 \cdot (-1)^{k+1}}{2k-1} \qquad S = \sum_{k=1}^{\infty} \frac{4 \cdot (-1)^{k+1}}{2k-1} = \pi$$

Typing this into Desmos we get the sigma notation by typing "sum".

Practice: Write out the terms of the following sums and compute the sum

$$\sum_{k=1}^{5} 3k \qquad \sum_{n=1}^{4} \frac{3n+2}{n+1} \qquad \sum_{i=0}^{3} 2^i$$

$= 3 + 6 + 9 + 12 + 15$

$= 45$

$= \frac{5}{2} + \frac{8}{3} + \frac{11}{4} + \frac{14}{5}$

$= \frac{643}{60}$

$= 1 + 2 + 4 + 8$

$= 15$

Practice: Write the following series in sigma notation:

$r = \frac{24}{8} = 3$

$$8 + 24 + 72 + 216 + 648$$

$a_k = 8 \cdot 3^{k-1}$

$\sum_{k=1}^{5} 8 \cdot 3^{k-1} = \sum_{k=0}^{4} 8 \cdot 3^{k}$

$r = -3$

$$\frac{2}{9} - \frac{2}{3} + 2 - 6 + 18 - 54$$

$\sum_{k=1}^{6} \frac{2}{9}(-3)^{k-1} = \sum_{k=0}^{5} \frac{2}{9}(-3)^{k}$

Now we would like to consider the geometric series, that is

$$S_n = \sum_{k=1}^{n} a_1 \cdot r^{k-1}$$

Oftentimes it is helpful to try and add or subtract copies of the summation to reduce it to something simpler.

$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$

$r S_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \ldots + a_1 r^{n-1} + a_1 r^n$

$(1-r) S_n = a_1 - a_1 r^n \Rightarrow S_n = \frac{a_1(1-r^n)}{1-r}$

first term

common ratio

$A \cdot r^{k-1}$

Example: Determine the following sums

$$\sum_{k=1}^{10} 4 \cdot 3^{k-1}$$

$r = 3$
$a_1 = 4$

$= 4\left(\frac{1-3^{10}}{1-3}\right)$

$= 118096$

$$\sum_{k=0}^{5} 3 \cdot \left(\frac{1}{2}\right)^k$$

$= \frac{3(1-(1/2)^6)}{1-1/2}$

$= 5.90625$

Practice: Determine the following sums

$$\sum_{k=1}^{7} 8 \cdot \left(-\frac{2}{3}\right)^{k-1}$$

$$= \frac{8\left(1-(-2/3)^7\right)}{1+2/3}$$

$$= 5.0809\ldots$$

$$\sum_{k=0}^{6} \left(\frac{9}{4}\right)^{k}$$

$$= \frac{1-(9/4)^7}{1-9/4}$$

$$= 232.743\ldots$$

Since we know the finite sum, we can consider what S_∞ would be:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$n \to \infty$

if $r > 1$

$r^n \to \infty$

$2^{100} >> 0$

if $|r| < 1$

$r^n \to 0$

$0.5^{100} \sim 0$

$$S_\infty = \frac{a_1}{1-r}$$

Example: Determine the following sums

$$\sum_{k=1}^{\infty} 10 \cdot \left(-\frac{2}{5}\right)^{k-1}$$

$$S_\infty = \frac{10}{1-(-2/5)} = \frac{10}{7/5}$$

$$= \frac{50}{7}$$

$$\sum_{k=0}^{\infty} 7 \cdot \left(-\frac{3}{4}\right)^{k}$$

$$S_\infty = \frac{7}{1-(-3/4)} = 4$$

$$\sum_{k=2}^{\infty} 100 \cdot \left(\frac{1}{2}\right)^{k}$$

start $\frac{100}{4} = 25$

$$= \frac{25}{1-1/2} = 50$$

$$\sum_{k=0}^{\infty} 12 \cdot (1.01)^{k}$$

$$= \infty$$

$1.01 \geq 1$

$1.01^{200} = 7.316$

$1.01^{300} = 19.78\ldots$

Practice Problems: Handout sections I, II, III, V(a,c,d)
Sigma Notation Module:
@ blogs.ubc.ca/infiniteseriesmodule/units/unit-1/sigma-notation/
Ryerson pdf: 1.3 # 1-7, 23
1.4 # 1-8, 16-19
1.5 # 1-7, 20