Sequences and Series

KNOW	DO	UNDERSTAND
Sequences are functioning whose domain is discrete. Can identify if a sequence is geometric.	Can use sigma notation to describe series. Can build the equation for a geometric sequence. Can determine the sum of a geometric series.	None yet

Vocab & Notation

- Sequence, $(a_n)_{n\in\mathbb{N}}$
- Index
- Recursion
- Series
- Sigma Notation, $\sum k$
- Geometric
- Common Ratio

Definition: A **sequence** is an ordered collection of objects. Like a set, but order matters. It is denoted as

$$(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots)$$

term

The position of an element of a sequence is called the **index**, and there is a relationship between the index and the **term.**

$$s: \mathbb{N} \to \mathbb{R}$$
$$s: n \mapsto a_n$$

Examples:

$(2, 4, 6, 8, \dots)$ 10, 12 $(2n)_{n\in\mathbb{N}}$	$a_{n+1} = a_n + 2$ (recursive), $a_1 = 2$ $a_n = 2n$ (closed)	A005843
$(2, 4, 8, 16, \dots)$ 32, 64 … $(2^n)_{n\in\mathbb{N}}$	$a_{n+1} = 2a_n$, $a_1 = 2$ $a_n = 2^n$	A000079
$(2, 3, 5, 7, 11, \dots)$ 13, 17, 19 Primes	P_n is the n^{th} prime ~~1~~, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~	A000040
$(2, 3, 5, 8, 13, \dots)$ 21, 34, 55 Fibonacci	$a_{n+1} = a_{n-1} + a_n$, $a_1 = 2$ and $a_2 = 3$ $a_n \sim \varphi^n + \bar{\varphi}^n$ φ is golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$	A000045

There is an online encyclopedia of integer sequences called OEIS and found at "oeis.org". The sequence number is shown on the side.

Oftentimes it is helpful to define a sequence starting with index 0 instead of 1. Really any number can be used to start the index but 0 and 1 are the main choices.

same

$$\underset{\text{start}}{(2n)_{n\in\mathbb{N}}} = \underset{\text{start}}{(2n+2)_{n=0}^{\infty}} = (2(n+1))_{n=0}^{\infty} = (2k)_{k=1}^{\infty}$$

let $k = n+1$

Definition: A **geometric sequence** is a sequence generated by multiplying the previous term by a fixed value.

$$\frac{a_{k+1}}{a_k} = \frac{a_k \cdot r}{a_k}$$

Definition: The **common ratio** is the ratio of consecutive terms in a geometric sequence.

any ratio of consecutive terms $\rightarrow$

$$\frac{a_{k+1}}{a_k} = r$$

Let's look and see how a geometric sequence is built by letting $a_1 = A$

$$(a_k)_{k\in\mathbb{N}} = (A, A\cdot r, A\cdot r^2, A\cdot r^3, \dots, A\cdot r^{n-1}, \dots) = (Ar^{k-1})_{k\in\mathbb{N}}$$

$$(a_1, a_2, a_3, a_4, \dots, a_n$$

Example: Determine sequences of the form $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ for the following patterns:

$\frac{6}{2} = 3 = \frac{18}{6} = 3$ $(2, 6, 18, \cdots)$

$$(2\cdot 3^{n-1})_{n=1}^{\infty} = (2\cdot 3^k)_{k=0}^{\infty}$$

let $k = n-1$

if $n=1$ $k=0$

$(100, -50, 25, \cdots)$

$r = -\frac{1}{2}$

$$\left(100\cdot\left(-\frac{1}{2}\right)^{n-1}\right)_{n=1}^{\infty} = \left(\frac{100}{(-2)^{n-1}}\right)_{n=1}$$

$$= \left(100\cdot\left(-\frac{1}{2}\right)^k\right)_{k=0}^{\infty}$$

$\left(\frac{8}{9}, \frac{2}{9}, \frac{1}{18}, \cdots\right)$

$\frac{2}{9}/\frac{8}{9} = \frac{1}{4} = r$

$$\left(\frac{8}{9}\cdot\left(\frac{1}{4}\right)^{n-1}\right)_{n\in\mathbb{N}} = \left(\frac{8}{9}\cdot\frac{1}{4^k}\right)_{k=0}^{\infty}$$

$$= \left(\frac{8}{9}\cdot 4^{-k}\right)_{k=0}^{\infty}$$

$\left(-9, -25, -\frac{625}{9}, \cdots\right)$

$r = \frac{25}{9}$

Definition: A **series** is a sum of a sequence, either finite or infinite.

Example: Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_k = 4\cdot\frac{(-1)^{k+1}}{2k-1}$, then we have that the sequence is:

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{4}{1}, -\frac{4}{3}, \frac{4}{5}, -\frac{4}{7}, \frac{4}{9}, -\frac{4}{11}, \frac{4}{13}, \dots\right)$$

$a_k \rightarrow 0$ as $k\rightarrow\infty$

And we can add the first 5 terms or add all of them as

$$S_5 = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} = \frac{1052}{315} = 3.3396\ldots$$

$$S_\infty = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \pi \quad ?$$

We'll get to the last sum in a second, but there is a better way to express the summation using **sigma notation**

end, sigma

$$S_n = \sum_{\boxed{k=1}}^{n} a_k$$

index, index, start

start @ k=1

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$ ← stop end @ k=n

So the sums we were looking at can be written as

$$S_5 = \sum_{k=1}^{5} \frac{4\cdot(-1)^{k+1}}{2k-1} \qquad S = \sum_{k=1}^{\infty} \frac{4\cdot(-1)^{k+1}}{2k-1} = \pi$$

Typing this into Desmos we get the sigma notation by typing "sum".

Practice: Write out the terms of the following sums and compute the sum

$$\sum_{k=1}^{5} 3k \qquad \sum_{n=1}^{4} \frac{3n+2}{n+1} \qquad \sum_{i=0}^{3} 2^i$$

$= 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 45$

$= \frac{5}{2} + \frac{8}{3} + \frac{11}{4} + \frac{14}{5}$

$= \frac{643}{60}$

$= 1 + 2 + 4 + 8$

$= 15$

Practice: Write the following series in sigma notation:

$\frac{24}{8}=3$

$\frac{72}{24}=3$

$$8+24+72+216+648$$

$$\sum_{k=1}^{5} 8\cdot 3^{k-1}$$

$$\sum_{k=0}^{4} 8\cdot 3^{k}$$

$r=-3$

$$\frac{2}{9}-\frac{2}{3}+2-6+18-54$$

$$\sum_{k=1}^{6} \frac{2}{9}(-3)^{k-1}$$

$$\sum_{k=0}^{5} \frac{2}{9}(-3)^{k}$$

Now we would like to consider the geometric series, that is

$$S_n=\sum_{k=1}^{n} a_1\cdot r^{k-1}$$

Oftentimes it is helpful to try and add or subtract copies of the summation to reduce it to something simpler.

$$S_n = a_1 + a_1\cdot r + a_1r^2 + a_1r^3 + \dots + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^{n-1} + a_1r^n$$

$$(1-r)S_n = a_1 - a_1r^n$$

$$\Rightarrow S_n = a_1\frac{(1-r^n)}{1-r}$$

first term   common ratio

Example: Determine the following sums

$$\sum_{k=1}^{10} 4\cdot 3^{k-1}$$

$$S_{10}=4\frac{(1-3^{10})}{1-3}$$

$$=118\ 096$$

$$\sum_{k=0}^{5} 3\cdot\left(\frac{1}{2}\right)^{k}$$

$$S_6 = 3\frac{(1-(1/2)^6)}{1-1/2}$$

$$=5.90625$$

Practice: Determine the following sums

$$\sum_{k=1}^{7} 8 \cdot \left(-\frac{2}{3}\right)^{k-1}$$

$$S_7 = 8\left(\frac{1-(-2/3)^7}{1-(-2/3)}\right)$$

$$= 5.0809\ldots$$

$$\sum_{k=0}^{6} \left(\frac{9}{4}\right)^{k}$$

$$S_7 = 1\left(\frac{1-(9/4)^7}{1-9/4}\right)$$

$$= 232.743\ldots$$

Since we know the finite sum, we can consider what S_∞ would be:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Example: Determine the following sums

$$\sum_{k=1}^{\infty} 10 \cdot \left(-\frac{2}{5}\right)^{k-1}$$

$$\sum_{k=0}^{\infty} 7 \cdot \left(-\frac{3}{4}\right)^{k}$$

$$\sum_{k=2}^{\infty} 100 \cdot \left(\frac{1}{2}\right)^{k}$$

$$\sum_{k=0}^{\infty} 12 \cdot (1.01)^{k}$$

Practice Problems: Handout sections I, II, III, V(a,c,d)
Sigma Notation Module:
@ blogs.ubc.ca/infiniteseriesmodule/units/unit-1/sigma-notation/
Ryerson pdf: 1.3 # 1-7, 23
1.4 # 1-8, 16-19
1.5 # 1-7, 20